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CROSS CHARACTERISTIC REPRESENTATIONS OF EVEN CHARACTERISTIC SYMPLECTIC GROUPS

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Abstract. We classify the small irreducible representations of $Sp_{2n}(q)$ with q even in odd characteristic. This improves even the known results for complex representations. The smallest representation for this group is much larger than in the case when q is odd. This makes the problem much more difficult.

1. Introduction

In [LaS], Landazuri and Seitz gave lower bounds for irreducible representations of Chevalley groups in cross characteristic, and these lower bounds were improved further by Seitz and Zalesskii in [SZ]. The Landazuri-Seitz-Zalesskii bounds have proved to be useful in many applications. In further applications, particularly in various problems related to the classification of maximal subgroups of finite classical groups (cf. for instance [GPPS], [MT1], [MMT]), in the minimal polynomial problem (see e.g. [GMST]), in the determination of submodule structure of small rank permutation modules ([LST], [ST]), and in computer programs to recognize linear groups of moderate degree, it is also important to identify the modules which have dimension close to the smallest possible dimension and to prove that there are no irreducible modules with dimension in a certain range above it. This was done in [GPPS] and [GT] for $SL_n(q)$. Further improvements were obtained by Brundan and Kleshchev [BrK]. Hiss and Malle [HM] have obtained results similar to [GT] for unitary groups, and their results have recently been improved in [GMST]. The case of $Sp_{2n}(q)$ with odd q has recently been done in [GMST].

In this paper, we consider representations of the finite symplectic groups $G = Sp_{2n}(q)$ with $n \geq 2$ and q even over an algebraically closed field $\mathbb{F}$ of characteristic $\ell \neq 2$. Landazuri and Seitz [LaS] had already shown that the smallest dimension

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of nontrivial cross characteristic representations of G is

$$d(G) := (q^n - 1)(q^n - q)/2(q + 1).$$

Decomposition matrices are known in the case of $Sp_4(q)$ [Wh1], and in the case of unipotent blocks of $Sp_6(q)$ [Wh2].

One of the principal differences between the cases of $SL_n(q)$, $SU_n(q)$, and $Sp_{2n}(q)$ with odd q that have been treated before and the case we are considering here is that the minimum dimension of cross characteristic representations of $G = Sp_{2n}(q)$ with q even is not small (compare $d(G)$ to $(q^n - 1)/2$ which is the bound for $Sp_{2n}(q)$ when q is odd). Moreover, in all the previously-studied cases one has obvious candidates for small representations which are the well-known Weil modules (see e.g. [GMST]). In our case, even the complex irreducible representations of small degree of G are much less understood, and their construction is far from being obvious. (Notice that the classification of small complex representations as given in [TZ1] only went up to $d(G)(1 + O(q^{-n}))$.) It turns out, however, that in many respects they still resemble the Weil representations that occur in the previous cases. In fact, these representations of G come from Weil representations of $SL_{2n}(q)$, in which case we call them (complex) *linear-Weil representations* of G, or from Weil representations of $SU_{2n}(q)$, in which case we call them (complex) *unitary-Weil representations* of G. A good construction and a better understanding of these complex representations are furnished using the concept of Howe's dual pairs [Hw2]. This concept is fairly well-known for finite groups of Lie type in odd characteristic; however in the case of Lie type groups in characteristic 2 this has been worked out first in [T]. When reduced modulo ℓ, these complex Weil representations produce $((q-1)_{\ell'} + 3)/2$ irreducible *linear-Weil representations* and $((q+1)_{\ell'} + 3)/2$ irreducible *unitary-Weil representations* in characteristic ℓ. Formal definitions and properties of complex and modular Weil representations of G are described in §§3, 7. For the reader's convenience, we collect in Table I the relevant information about complex and modular Weil characters of $Sp_{2n}(q)$ (extracted from Corollaries 7.5 and 7.10). In this table, $\widehat{\chi}$ denotes the restriction of a character χ to ℓ'-elements, and $N_{\ell'}$ denotes the ℓ'-part of an integer N.

As in the case of odd q, Weil representations of G can also be characterized by certain local properties, $(\mathcal{W}_1)$ and $(\mathcal{W}_2^{\varepsilon})$, which are studied in §§3, 5. The most transparent one, $(\mathcal{W}_2^{\varepsilon})$, means that the representation in question does not afford a P_2-orbit of Z_2-characters of length $q(q-1)(q-\varepsilon)/2$ (which correspond to all quadratic forms of rank 2 and type $-\varepsilon$ on $\mathbb{F}_q^2$). Here P_2 is the stabilizer in G of a 2-dimensional totally isotropic subspace of the natural module $\mathbb{F}_q^{2n}$ of G, and $Z_2 = Z(O_2(P_2))$.

We will exploit the method that has been first developed in [GMST], which is to analyze modules with various local properties, and by restricting to various families of subgroups which contain a conjugate of every element of the group, we can determine the Brauer character of the module in question. However, another principal difference between our case and the case of $Sp_{2n}(q)$ with odd q considered in [GMST] is that G contains some semisimple elements (namely elements of tori of order $q^n \pm 1$) which cannot be covered by subgroups good enough for the purposes of this "gluing" method. To cover those elements we rely heavily upon Deligne-Lusztig theory [L], and upon some fundamental results of Broué and Michel [BM] on unions of ℓ-blocks and of Geck and Hiss [GH] on basic sets of Brauer characters, which also depend on Deligne-Lusztig theory.

TABLE I. Weil characters of $Sp_{2n}(q)$ (q even, $n \geq 2$, $\ell \neq 2$)

Complex linear-Weil characters	Degree	ℓ-modular linear-Weil characters
ρ_n^1	$\dfrac{(q^n+1)(q^n-q)}{2(q-1)}$	$\hat{\rho}_n^1 - \begin{cases} 1, & \ell \mid \frac{q^n-1}{q-1}, \\ 0, & \text{otherwise} \end{cases}$
ρ_n^2	$\dfrac{(q^n-1)(q^n+q)}{2(q-1)}$	$\hat{\rho}_n^2 - \begin{cases} 1, & \ell \mid (q^n+1), \\ 0, & \text{otherwise} \end{cases}$
τ_n^i, $1 \leq i \leq (q-2)/2$	$\dfrac{q^{2n}-1}{q-1}$	$\hat{\tau}_n^i$, $1 \leq i \leq ((q-1)_{\ell'}-1)/2$
Complex unitary-Weil characters	Degree	ℓ-modular unitary-Weil characters
α_n	$\dfrac{(q^n-1)(q^n-q)}{2(q+1)}$	$\widehat{\alpha}_n$
β_n	$\dfrac{(q^n+1)(q^n+q)}{2(q+1)}$	$\widehat{\beta}_n - \begin{cases} 1, & \ell \mid (q+1), \\ 0, & \text{otherwise} \end{cases}$
ζ_n^i, $1 \leq i \leq q/2$	$\dfrac{q^{2n}-1}{q+1}$	$\hat{\zeta}_n^i$, $1 \leq i \leq ((q+1)_{\ell'}-1)/2$

The main results of the paper are the following two theorems. In Theorem 1.1 we define $\alpha = 19/15$ if $(n, q) = (5, 2)$, $\alpha = 2$ if $(n, q) = (5, 4)$ or $(6, 2)$, and $\alpha = 0$ otherwise.

Theorem 1.1. *Let $G = Sp_{2n}(q)$ with q even and $n \geq 2$. Let V be an absolutely irreducible G-module in characteristic $\ell \neq 2$ of dimension less than*

$$\mathfrak{d}(n,q) := \begin{cases} q^2(q-1), & n = 2, \\ 21, & (n,q) = (3,2), \\ q^2(q^3-1), & n = 3,\ q > 2, \\ 203, & (n,q) = (4,2), \\ (q^4-1)(q^3-1)q^2, & n = 4,\ q > 2, \\ \left(\frac{(q^{n-1}+1)(q^{n-2}-q)}{q^2-1} - 1 - \alpha\right) \frac{q^{n-1}(q^{n-1}-1)(q-1)}{2}, & n \geq 5. \end{cases}$$

Then V is either the trivial module or a Weil module.

Theorem 1.2. *Let $G = Sp_{2n}(q)$, q be even, $n \geq 2$, $(n, q) \neq (2, 2)$, $(3, 2)$. Let V be an absolutely irreducible G-module in characteristic $\ell \neq 2$ that satisfies $(\mathcal{W}_2^\varepsilon)$ for some $\varepsilon = \pm$. Then one of the following holds:*

(i) *V is the trivial module.*

(ii) *$\varepsilon = +$ and V is a linear-Weil module.*

(iii) *$\varepsilon = -$ and V is a unitary-Weil module.*

Observe that if $n \geq 5$, then $\mathfrak{d}(n, q) = \frac{1}{2}q^{4n-6}(1-q^{-1}+O(q^{-2}))$. In the meantime, Corollary 6.2 shows that G has a complex irreducible non-Weil character of degree $D(n, q) = \frac{1}{2}q^{4n-6}(1 + q^{-4} + O(q^{-5}))$. Hence the bound $\mathfrak{d}(n, q)$ given in Theorem 1.1 is the asymptotically correct bound when $n \geq 5$. We emphasize that Theorem 1.1 is proved relying upon Theorem 1.2. Moreover, Theorem 1.2 is also used in applications to identify the representations V in question via the local property $(\mathcal{W}_2^\varepsilon)$, when a priori no upper bound on $\dim(V)$ is given. The results of the paper have already been used in [MT2] to determine irreducible tensor products of cross characteristic representations of $Sp_{2n}(q)$ with q even.

The orthogonal groups will be treated in a forthcoming paper.

Notation. Throughout the paper, q is a power of 2, $G = Sp_{2n}(q)$, $\mathbb{F}$ is an algebraically closed field of characteristic $\ell \neq 2$, I_n is the $n \times n$ identity matrix, and $J_n = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$. Furthermore, $M = \mathbb{F}_q^{2n}$ is equipped with a G-invariant nondegenerate symplectic form $(\cdot,\cdot)$, which has J_n as Gram matrix in a fixed basis $(e_1, \ldots, e_n, f_1, \ldots, f_n)$. By a standard subgroup $Sp_{2j}(q)$ in G we mean the pointwise stabilizer of a nondegenerate $(2n-2j)$-dimensional subspace of M. Also, P_j is the stabilizer of a j-dimensional totally isotropic subspace in M, $Q_j = O_p(P_j)$, $Z_j = Z(Q_j)$ if $j > 1$ and $Z_1 = Z(P_1')$, and $L_j \simeq Sp_{2n-2j}(q) \times GL_j(q)$ is the Levi subgroup. Next, $H_d \simeq Sp_{2d}(q) \times Sp_{2n-2d}(q)$ is the stabilizer of a $2d$-dimensional nondegenerate subspace of M. Let $M_{m,n}(q)$ be the set of $m \times n$-matrices over $\mathbb{F}_q$, $M_n(q) = M_{n,n}(q)$, $\mathcal{H}_n(q) = \{X \in M_n(q) \mid {}^tX + X = 0\}$, $\mathcal{H}_n^0(q) = \{X \in \mathcal{H}_n(q) \mid X$ has zero diagonal$\}$, $\mathcal{F}_n(q) = M_n(q)/\mathcal{H}_n(q)$. If χ is a class function of G, then $\widehat{\chi}$ is the restriction of χ to ℓ'-elements. The symbol $GL_n^\alpha(q)$ stands for $GL_n(q)$ if $\alpha = +$ and $GU_n(q)$ if $\alpha = -$. We fix a primitive $(q-1)^{\text{th}}$ root δ of unity in $\mathbb{F}_q$, and a primitive $(q-1)^{\text{th}}$ root $\tilde{\delta}$ of unity in $\mathbb{C}$. Furthermore, we fix a primitive $(q+1)^{\text{th}}$ root ξ of unity in $\mathbb{F}_{q^2}$, and a primitive $(q+1)^{\text{th}}$ root $\tilde{\xi}$ of unity in $\mathbb{C}$. If N is an integer and r is a prime, then $N_{r'}$ denotes the r'-part of N. If N is a prime power, then $[N]$ denotes the elementary abelian group of order N.

All Brauer characters are meant to be in characteristic ℓ. If A, B, C are $\mathbb{F}X$-modules for a group X, then we will use the notation $C = A + B$ to indicate that this is true in the Grothendieck group $G_0(X)$. Abusing notation, sometimes we will denote an $\mathbb{F}X$-module and its Brauer character by the same letter.

2. Preliminaries

We will frequently use the following version of [GMST, Lemma 4.2]:

Lemma 2.1. *Let V be an $\mathbb{F}G$-module with $C_V(G) = 0$ and let U be an L_1-composition factor of $C_V(Q_1)$. Then U is also an L_1-composition factor of $[V, Q_1]$.*

Proof. We may assume that $P_1 = Stab_G(\langle e_1 \rangle_{\mathbb{F}_q})$. Let $g \in G$ be the element that flips e_1 and f_1 and fixes all e_i and f_i with $i \geq 2$. Then $L_1 = Stab_G(\langle e_1 \rangle_{\mathbb{F}_q}, \langle f_1 \rangle_{\mathbb{F}_q})$ normalizes both Q_1 and Q_1^g. Thus $N_G(\langle Q_1, Q_1^g \rangle)$ contains $\langle Q_1, Q_1^g \rangle L_1$ and therefore equals G; in other words $\langle Q_1, Q_1^g \rangle \lhd G$. But G is simple, hence $\langle Q_1, Q_1^g \rangle = G$. Notice also that g centralizes L_1. Therefore, our statement follows from [GMST, Lemma 4.2]. $\square$

The next result gives families of subgroups which contain a conjugate of every element in G. If $kl = n$, then $Sp_{2k}(q^l)$ naturally embeds in G by viewing the natural $2k$-dimensional module over $\mathbb{F}_{q^l}$ as a $2n$-dimensional vector space over $\mathbb{F}_q$. For each $\varepsilon = \pm$, $T_\varepsilon \simeq \mathbb{Z}_{q^n - \varepsilon}$ is a torus of $SL_2(q^n)$ naturally embedded in G.

Lemma 2.2. *Let $g \in G = Sp_{2n}(q)$ with $n \geq 3$. Then a G-conjugate of g is contained in at least one of the following subgroups:*

(i) *P_j with $1 \leq j \leq n-1$, H_d with $1 \leq d \leq n/2$, and $Sp_4(q^{n/2})$, provided that n is even;*

(ii) *P_j with $1 \le j \le n-1$, H_d with $1 \le d \le n/2$, and $Sp_{2k}(q^{n/k})$, provided that n is odd and divisible by $k > 1$;*

(iii) *P_j with $1 \le j \le n-1$, H_d with $1 \le d \le n/2$, T_+, and T_-, provided that n is an odd prime.*

Proof. 1) First consider the case when a conjugate of g is contained in P_n. We claim that either a conjugate of g is contained in some P_i with $1 \le i \le n-1$, or g is conjugate to an element of T_+, or n is even and a conjugate of g is contained in $Sp_4(q^{n/2})$. Indeed, we may assume that g stabilizes $U := \langle e_1, \ldots, e_n \rangle_{\mathbb{F}_q}$. Let $g = su$, where s is the semisimple part and u is the unipotent part of g. Since u is a power of g, $u(U) = U$. If $u|_U \neq 1_U$, then g fixes $C_U(u)$, which is clearly a nonzero totally singular subspace and of dimension $j < n$, whence a conjugate of g is contained in P_j. Assume that $u = 1$ on U. If $s|_U$ is not irreducible, then s (and g as well, since $u = 1$ on U) fixes a proper subspace U' of U, and so a conjugate of g is contained in $P_{j'}$ with $j' = \dim(U') < n$. Assume $s|_U$ is irreducible. Then $s|_U$ is conjugate (in the algebraic group $SL_n(\overline{\mathbb{F}}_q)$) to $\mathrm{diag}(\sigma, \sigma^2, \ldots, \sigma^{q^{n-1}})$ with $\sigma^{q^n-1} = 1$ but $\sigma^{q^k} \neq \sigma$ for $0 < k < n$. If the s-module U is not self-dual, then $C_G(s)$ is conjugate to T_+ and so a conjugate of g is contained in T_+. Assume the s-module U is self-dual. In this case there must be some k, $0 \le k < n$, such that $\sigma^{q^k} = \sigma^{-1}$. It follows that n is even, $C_G(s) \simeq GU_2(q^{n/2}) < Sp_4(q^{n/2})$, whence g is conjugate to an element of $Sp_4(q^{n/2})$.

2) By [GMST, Lemma 4.4], a conjugate of any $g \in G$ is contained in some P_i, H_d, or $SL_2(q^n)$. Observe that if $x \in SL_2(q^n)$, then either $x \in P_n$, or $x \in T_-$. Hence a conjugate of any $g \in G$ is contained in some P_i, H_d, or T_-.

3) Assume n is even. Then we may embed $T_\pm \hookrightarrow SL_2(q^n) \hookrightarrow Sp_4(q^{n/2})$. Hence the results of 1) and 2) yield statement (i).

Assume n is odd and divisible by some $k > 1$. Then we may embed $T_\pm \hookrightarrow SL_2(q^n) \hookrightarrow Sp_{2k}(q^{n/k})$. Hence the results of 1) and 2) yield statement (ii).

Finally, 1) and 2) also imply statement (iii) in the case when n is an odd prime. $\square$

Let X be any finite group with a normal ℓ'-subgroup Q. If V is any $\mathbb{F}X$-module and $\lambda \in \mathrm{IBr}_\ell(Q)$, then we will denote by V_λ the λ-homogeneous component of $V|_Q$, and let $I_\lambda := Stab_X(\lambda)$. Next, if φ is a virtual Brauer character of X, then we may assume that φ is afforded by $V - U$, where V and U are $\mathbb{F}X$-modules. In this case we may consider the virtual Brauer character of I_λ afforded by $V_\lambda - U_\lambda$. This character does not depend on the choice of V, U by the following lemma, hence we may talk about the virtual I_λ-module $(V-U)_\lambda := V_\lambda - U_\lambda$ and call it the λ-homogeneous component of φ.

Lemma 2.3. *Let U, V, U', V' be $\mathbb{F}X$-modules such that $V - U = V' - U'$. Then the virtual I_λ-characters afforded by $V_\lambda - U_\lambda$ and by $V'_\lambda - U'_\lambda$ are equal.*

Proof. It suffices to show that the virtual Brauer characters in question take the same value at any ℓ'-element $g \in I_\lambda$. Replacing X by $Y := Q\langle g \rangle$, we may consider U, V, U', V' as complex Y-modules. By assumption, the complex Y-modules $V + U'$ and $V' + U$ have same character, whence they are equivalent. It follows that their λ-homogeneous components $(V+U')_\lambda = V_\lambda + U'_\lambda$ and $(V'+U)_\lambda = V'_\lambda + U_\lambda$ are equivalent. Thus the traces of g acting on $V_\lambda + U'_\lambda$ and $V'_\lambda + U_\lambda$ are equal. Consequently, the virtual traces of g on $V_\lambda - U_\lambda$ and $V'_\lambda - U'_\lambda$ are equal, as stated. $\square$

We will also need the following statement:

Lemma 2.4. *Let r be a prime and let Q be a normal extraspecial r-subgroup of order r^{1+2n} of a finite group X. Suppose that χ is an irreducible complex character of X of degree r^n such that $\chi|_Q \in \mathrm{Irr}(Q)$. Then for any $g \in X$, $|\chi(g)|^2 = |C_{Q/Z(Q)}(g)|$ if g acts trivially on the complete inverse image of $C_{Q/Z(Q)}(g)$ in Q, and $\chi(g) = 0$ otherwise.*

Proof. The statement is well known; cf. for instance [Is, Theorem (3.5)]. Even though it was assumed in [Is] that $r > 2$, the argument given there goes through for any r. For completeness we give an outline of the argument. Denote $Z := Z(Q)$, $\bar{C} := C_{Q/Z}(g)$, and let C be the complete inverse image of $\bar{C}$ in Q. It is clear that $\chi|_Z = r^n\lambda$ for some faithful linear character $\lambda \in \mathrm{Irr}(Z)$, and that $Z = Z(X)$. First assume g does not act trivially on C. Then there are some $x \in Q$ and $1 \neq z \in Z$ such that $xgx^{-1} = gz$, whence $\chi(g) = \chi(g)\lambda(z)$ and so $\chi(g) = 0$. Next, assume g acts trivially on C. Consider the alternating form $(xZ, yZ) \mapsto [x, y] \in Z$ for $xZ, yZ \in Q/Z$. Since this form is nondegenerate, one can check that the orthogonal complement $\bar{C}^{\perp}$ is exactly $[g, Q/Z]$. If $u \in Q$, then $C_{Q/Z}(gu) = \bar{C}$; moreover, gu acts trivially on C if and only if u centralizes C, i.e. $uZ \in \bar{C}^{\perp}$. It follows that the coset gQ contains exactly $|Z| \cdot |\bar{C}^{\perp}|$ elements gu that act trivially on C. For each such element gu, we have already shown that $uZ \in \bar{C}^{\perp} = [g, Q/Z]$, therefore $g^{-1}vgv^{-1} = uz$ for some $v \in Q$ and $z \in Z$, i.e. $vgv^{-1} = guz$, whence $|\chi(gu)| = |\chi(guz)| = |\chi(g)|$. Thus $\sum_{x \in gQ} |\chi(x)|^2 = \sum_{uZ \in \bar{C}^{\perp}} |\chi(gu)|^2 = |Z| \cdot |\bar{C}^{\perp}| \cdot |\chi(g)|^2$. On the other hand, $\sum_{x \in gQ} |\chi(x)|^2 = |Q|$ by [Is, Lemma (3.4)]. Consequently, $|\chi(g)|^2 = |Q/Z|/|\bar{C}^{\perp}| = |\bar{C}|$, as stated. □

3. Local properties $(\mathcal{W}_2^{\varepsilon})$ and Weil representations

First we make some observations about the structure and representations of $P_j = Stab_G(\langle e_1, \ldots, e_j \rangle_{\mathbb{F}_q})$ for $1 \leq j \leq n$. It is convenient to write elements of P_j with respect to the basis

$$(e_1, \ldots, e_j, e_{j+1}, \ldots, e_n, f_{j+1}, \ldots, f_n, f_1, \ldots, f_j).$$

For any $A \in M_{2n-2j,j}(q)$ and $C \in M_j(q)$, set $[A, C] := \begin{pmatrix} I_j & {}^tAJ_{n-j} & C \\ 0 & I_{2n-2j} & A \\ 0 & 0 & I_j \end{pmatrix}$. Then

$$Q_j = \left\{[A, C] \mid A \in M_{2n-2j,j}(\mathbb{F}_q),\ C \in M_j(q),\ C + {}^tC + {}^tAJ_{n-j}A = 0\right\}$$

has order $q^{j(4n-3j+1)/2}$. The multiplication in Q_j is given by the formula

$$[A, C] \cdot [A', C'] = [A + A', C + C' + {}^tAJ_{n-j}A'].$$

In particular, $Z_j = \{[0, C] \mid C \in \mathcal{H}_j(q)\}$ and $Q'_j = \{[0, C] \mid C \in \mathcal{H}_j^0(q)\}$ if $j > 1$.

Assume $j > 1$. Any linear character λ of Z_j (over $\mathbb{F}$) can be written in the form $\lambda_B\ :\ [0, X] \mapsto (-1)^{\mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_2}(\mathrm{Tr}(BX))}$ for some $B \in M_j(q)$. Clearly, $\lambda_B = \lambda_{B'}$ if and only if $B - B' \in \mathcal{H}_j^0(q)$, so we may think of the subscript B of λ_B as a coset in $\mathcal{F}_j(q)$. On the other hand, for each $B = (b_{ij}) \in M_j(q)$ we define the quadratic form q_B on the space $\mathbb{F}_q^j = \langle f_1, \ldots, f_j \rangle_{\mathbb{F}_q}$ such that $q_B(f_i) = b_{ii}$ and the associated bilinear form has $B + {}^tB$ as Gram matrix in the given basis. Again, $q_B = q_{B'}$ if and only if $B - B' \in \mathcal{H}_j^0(q)$, so we may think of the subscript B of q_B as a coset in $\mathcal{F}_j(q)$. Furthermore, for any $X \in GL_j(q)$, $\lambda_{{}^tXBX} = \lambda_B$ if and only if ${}^tXBX - B \in \mathcal{H}_j^0(q)$, i.e. ${}^tXBX$ and B have the same diagonal and ${}^tX({}^tB+B)X = {}^tB+B$, that is, $X \in O(q_B)$. Recall that the rank of q_B is the codimension of $\mathrm{rad}(q_B)$ in $\mathbb{F}_q^j$. Then two quadratic forms on $\mathbb{F}_q^j$ are $GL_j(q)$-equivalent if and only if they have the same rank and the same type if the rank is even. Let $\mathcal{O}_r^\epsilon$ be the set of all λ_B where q_B has rank r and type ϵ. Here $\epsilon = \pm$ if r is even and ϵ is void if r is odd. To ease the notation in what follows, we will also use the symbol $\mathcal{O}_r^\pm$ for $\mathcal{O}_r$ when r is odd. The above discussion yields the following statement:

Lemma 3.1. *Assume $1 < j \leq n$. Then the P_j-orbits on $\mathrm{IBr}_\ell(Z_j)$ are precisely $\mathcal{O}_r^\epsilon$ with $0 \leq r \leq j$. If $\lambda = \lambda_B$ belongs to $\mathcal{O}_r^\epsilon$, then $Stab_{L_j}(\lambda_B)$ is isomorphic to*

$$Sp_{2n-2j}(q) \times O(q_B) \simeq Sp_{2n-2j}(q) \times \left([q^{r(j-r)}] : (GL_{j-r}(q) \times O_r^\epsilon(q))\right). \quad \square$$

Definition 3.2. Let r be even, $2 \leq r \leq n$, $\varepsilon = \pm$.

(i) Let $r \leq j \leq n$. We say that an $\mathbb{F}P_j$-module V has property $(\mathcal{W}_r^\varepsilon)$ if all Z_j-characters that occur in the module $V|_{Z_j}$ belong to $\mathcal{O}_r^\varepsilon \cup \left(\bigcup_{0\leq s\leq r-1,\ \alpha=\pm} \mathcal{O}_s^\alpha\right)$.

(ii) We say that an $\mathbb{F}G$-module V has property $(\mathcal{W}_r^\varepsilon)$ if the $\mathbb{F}P_j$-module V satisfies $(\mathcal{W}_r^\varepsilon)$ for some $j \geq r$.

Definition 3.3. Let $2 \leq r \leq n$.

(i) Let $r < j \leq n$. We say that an $\mathbb{F}P_j$-module V has property $(\mathcal{W}_r)$ if all Z_j-characters that occur in the module $V|_{Z_j}$ belong to $\bigcup_{0\leq s\leq r,\ \alpha=\pm} \mathcal{O}_s^\alpha$.

(ii) We say that an $\mathbb{F}G$-module V has property $(\mathcal{W}_r)$ if the $\mathbb{F}P_j$-module V satisfies $(\mathcal{W}_r)$ for some $j > r$.

We say that a Brauer character φ has property $(\mathcal{W}_r^\varepsilon)$, resp. $(\mathcal{W}_r)$, if the same holds for a module affording this character.

Lemma 3.4. *Let V be an $\mathbb{F}G$-module.*

(i) *Suppose that $r \geq 2$ is even and the $\mathbb{F}P_j$-module V satisfies $(\mathcal{W}_r^\varepsilon)$ for some $j \geq r$. Then the $\mathbb{F}P_m$-module V satisfies $(\mathcal{W}_r^\varepsilon)$ for every $m \geq r$.*

(ii) *Suppose that $r \geq 2$ and the $\mathbb{F}P_j$-module V satisfies $(\mathcal{W}_r)$ for some $j > r$. Then the $\mathbb{F}P_m$-module V satisfies $(\mathcal{W}_r)$ for every $m > r$.*

Proof. (i) We may assume that $P_m = Stab_G(\langle e_1, \dots, e_m\rangle_{\mathbb{F}_q})$. Then

$$Z_m = Stab_G(e_1, \dots, e_n, f_{m+1}, \dots, f_n).$$

In particular, $Z_1 < Z_2 < \ldots < Z_n$.

First we show that the $\mathbb{F}P_n$-module V satisfies $(\mathcal{W}_r^\varepsilon)$. Assume the contrary: $V|_{Z_n}$ affords a character λ_B lying outside of $\mathcal{O}_r^\varepsilon \cup \left(\bigcup_{0\leq s\leq r-1,\ \alpha=\pm} \mathcal{O}_s^\alpha\right)$. Conjugating λ_B using L_n we may assume that $\mathrm{rad}(q_B) = \langle f_{s+1}, \dots, f_n\rangle_{\mathbb{F}_q}$, where $s = \mathrm{rank}(q_B)$. Observe that either $s > r$, or $s = r$ but q_B has type $-\varepsilon$. In either case we may choose $f_1, \dots, f_r$ such that q_B is nondegenerate and of type $-\varepsilon$ on $\langle f_1, \dots, f_r\rangle_{\mathbb{F}_q}$. It follows that q_B restricted to $\langle f_1, \dots, f_j\rangle_{\mathbb{F}_q}$ either has rank $> r$ or has rank r but not type ε. Hence, $\lambda_B|_{Z_j}$ cannot belong to $\mathcal{O}_r^\varepsilon \cup \left(\bigcup_{0\leq s\leq r-1,\ \alpha=\pm} \mathcal{O}_s^\alpha\right)$, a contradiction.

Now we may assume that for any λ_B occurring in $V|_{Z_n}$, either $\mathrm{rank}(q_B) < r$ or $\mathrm{rank}(q_B) = r$ but q_B has type ε (on $\langle f_1, \dots, f_n\rangle_{\mathbb{F}_q}$). The same obviously holds for the restriction of q_B to $\langle f_1, \dots, f_m\rangle_{\mathbb{F}_q}$ for any $m \geq r$. Hence $\lambda_B|_{Z_m}$ belongs to $\mathcal{O}_r^\varepsilon \cup \left(\bigcup_{0\leq s\leq r-1,\ \alpha=\pm} \mathcal{O}_s^\alpha\right)$, as desired.

(ii) Similarly to (i). $\square$

We will frequently use the following consequence of Lemma 3.4:

Corollary 3.5. *Let $G = Sp_{2n}(q)$ and let $H \simeq Sp_{2m}(q)$ be a standard subgroup of G, where $2 \leq m \leq n$. Assume $\varphi \in \mathrm{IBr}_\ell(G)$ and $\varepsilon = \pm$. Then φ satisfies $(\mathcal{W}_2^\varepsilon)$ if and only if $(\mathcal{W}_2^\varepsilon)$ holds for $\varphi|_H$ (equivalently, if $(\mathcal{W}_2^\varepsilon)$ holds for all irreducible constituents of $\varphi|_H$).*

Proof. We may assume that $H = Stab_G(e_{m+1}, \dots, e_n, f_{m+1}, \dots, f_n)$. Consider the parabolic subgroup $P_2(H) := Stab_H(\langle e_1, e_2\rangle_{\mathbb{F}_q})$ of H. Then its subgroup $Z_2(H) := O_2(P_2(H))$ is exactly the subgroup Z_2 for the parabolic subgroup $P_2 = Stab_G(\langle e_1, e_2\rangle_{\mathbb{F}_q})$ of G. By definition, φ satisfies $(\mathcal{W}_2^\varepsilon)$ if and only if all Z_2-characters occuring in $\varphi|_{Z_2}$ belong to $\mathcal{O}_0 \cup \mathcal{O}_1 \cup \mathcal{O}_2^\varepsilon$. Hence the statement follows. □

The properties $(\mathcal{W}_2^+)$ and $(\mathcal{W}_2^-)$ will play a central role throughout the paper. It turns out that they are the local properties that distinguish low-dimensional representations from the rest. Unlike the case of $SL_n(q)$, $SU_n(q)$ and the case of $Sp_{2n}(q)$ with q odd, where *Weil representations* have degree $< q^n$ and they are the obvious candidates for low-dimensional representations, $G = Sp_{2n}(q)$ with q even does not have any nontrivial representations of dimension less than $d(G) := (q^n-1)(q^n-q)/2(q+1)$ [LaS]. However, we can still introduce two classes of irreducible complex representations of G, which come from Weil representations of $SL_{2n}(q)$ and $SU_{2n}(q)$ and which we will call *Weil representations* of G.

We briefly recall the definition of complex Weil representations of $SL_{2n}(q)$. Consider the permutation character $\tau_n \;:\; g \mapsto |C_M(g)| = q^{\dim_{\mathbb{F}_q} \mathrm{Ker}(g-1)}$ of $SL_{2n}(q)$ on the points of the natural module $M := \mathbb{F}_q^{2n}$. It decomposes into irreducible constituents as $2 \cdot 1 + \sum_{i=0}^{q-2} \tau_n^i$, where

$$\tau_n^i(g) = \frac{1}{q-1} \sum_{j=0}^{q-2} \tilde{\delta}^{ij} q^{\dim_{\mathbb{F}_q} \mathrm{Ker}(g-\delta^j)} - 2\delta_{i,0} \tag{1}$$

for any $g \in SL_{2n}(q)$ (and $\mathrm{Ker}(g-\delta^j)$ is computed on M). The τ_n^i, $0 \leq i \leq q-2$, are the (irreducible) complex Weil representations of $SL_{2n}(q)$ (regardless if q is even or odd); cf. [Ge], [Hw1]. Formula (1) can be derived following the proof of [TZ2, Lemma 4.1].

We can embed $G = Sp_{2n}(q)$ in $SL_{2n}(q)$. Since the G-module M is self-dual, $\dim \mathrm{Ker}(g-\delta^j) = \dim \mathrm{Ker}(g-\delta^{-j})$ for any g and j. It follows that

$$\tau_n^i|_G = \tau_n^{q-1-i}|_G \text{ for } 1 \leq i \leq q-2. \tag{2}$$

It is known, see for instance [T], that the characters $\tau_n^i|_G$ with $1 \leq i \leq q/2-1$ are distinct irreducible characters of G (of degree $(q^{2n}-1)/(q-1)$), and we denote by the same symbol τ_n^i. On the other hand,

$$\tau_n^0|_G = \rho_n^1 + \rho_n^2, \tag{3}$$

where ρ_n^1, resp. ρ_n^2, is an irreducible character of G of degree $(q^n+1)(q^n-q)/2(q-1)$, resp. $(q^n-1)(q^n+q)/2(q-1)$.

Next we briefly recall the definition of complex Weil representations of $SU_{2n}(q)$. Consider the class function $\zeta_n : g \mapsto (-q)^{\dim_{\mathbb{F}_{q^2}} \mathrm{Ker}(g-1)}$ of $SU_{2n}(q)$ (where $\mathrm{Ker}(g-1)$ is the fixed point subspace of g on the natural module $\widetilde{M} := \mathbb{F}_{q^2}^{2n}$). One can prove (cf. [TZ2]) that ζ_n decomposes into irreducible constituents as $\sum_{i=0}^{q} \zeta_n^i$, where

$$\zeta_n^i(g) = \frac{1}{q+1} \sum_{j=0}^{q} \tilde{\xi}^{ij} (-q)^{\dim_{\mathbb{F}_{q^2}} \mathrm{Ker}(g-\xi^j)} \tag{4}$$

for any $g \in SU_{2n}(q)$ (and $\mathrm{Ker}(g-\xi^j)$ is calculated on $\widetilde{M}$). The ζ_n^i, $0 \leq i \leq q$ are the (irreducible) complex Weil representations of $SU_{2n}(q)$ (regardless if q is even or odd); cf. [Ge], [Hw1], [S].

We can identify $\widetilde{M}$ with $M \otimes_{\mathbb{F}_q} \mathbb{F}_{q^2}$ and embed $G = Sp_{2n}(q)$ in $SU_{2n}(q)$ (cf. for instance [KLi]). Since the G-module M is self-dual, it follows that $\dim \mathrm{Ker}(g-\xi^j) = \dim \mathrm{Ker}(g-\xi^{-j})$ for any g and j. It follows that

$$\zeta_n^i|_G = \zeta_n^{q+1-i}|_G \text{ for } 1 \leq i \leq q. \tag{5}$$

As shown in [T], the characters $\zeta_n^i|_G$ with $1 \leq i \leq q/2$ are distinct irreducible characters of G (of degree $(q^{2n}-1)/(q+1)$), and we denote by the same symbol ζ_n^i. On the other hand,

$$\zeta_n^0|_G = \alpha_n + \beta_n, \tag{6}$$

where α_n, resp. β_n, is an irreducible character of G of degree $(q^n-1)(q^n-q)/2(q+1)$, resp. $(q^n+1)(q^n+q)/2(q+1)$. Observe that $\alpha_n (\mathrm{mod}\, \ell)$ is irreducible, since its degree meets the Landazuri-Seitz-Zalesskii bound $d(G)$.

Definition 3.6. (i) The $q/2+1$ characters ρ_n^1, ρ_n^2, and τ_n^i with $1 \leq i \leq q/2-1$ are called (complex) *linear-Weil characters* of G. The nontrivial irreducible constituents of reductions modulo ℓ of these $q/2+1$ characters are called *linear-Weil characters* of G in characteristic ℓ.

(ii) The $q/2+2$ characters α_n, β_n, and ζ_n^i with $1 \leq i \leq q/2$ are called *unitary-Weil characters* of G. The nontrivial irreducible constituents of reductions modulo ℓ of these $q/2+2$ characters are called *unitary-Weil characters* of G in characteristic ℓ.

(iii) In general, a *Weil character* of G is either a linear-Weil or a unitary-Weil (Brauer) character.

Weil characters behave well when restricted to certain naturally embedded subgroups:

Lemma 3.7. *Let $G = Sp_{2n}(q)$ and let ρ be a Weil character of G. Let $H \simeq Sp_{2m}(q)$ be a standard subgroup of G, and let $K \simeq Sp_{2k}(q^l)$, where $kl = n$, be naturally embedded in G. Let ϕ be any nontrivial irreducible constituent of $\rho|_H$ and let ψ be any nontrivial irreducible constituent of $\rho|_K$.*

(i) *Assume that ρ is a linear-Weil character. Then ϕ, resp. ψ, is also a linear-Weil character.*

(ii) *Assume that ρ is a unitary-Weil character. Then ϕ is also a unitary-Weil character. If l is even, then ψ is linear-Weil, and if l is odd, then ψ is unitary-Weil.*

Proof. Clearly, $\tau_n|_H = q^{2n-2m}\tau_m$ and $\zeta_n|_H = q^{2n-2m}\zeta_m$, whence the statements follow for ϕ. Next, consider the characters $\tau_{k,q^l} : g \mapsto (q^l)^{\dim_{\mathbb{F}_{q^l}} \mathrm{Ker}(g-1)}$ and $\zeta_{k,q^l} : g \mapsto (-q^l)^{\dim_{\mathbb{F}_{q^{2l}}} \mathrm{Ker}(g-1)}$ of K, which play the roles that τ_n and ζ_n play for G. Then $\tau_n|_K = \tau_{k,q^l}$, and $\zeta_n|_K = \tau_{k,q^l}$ if l is even and ζ_{k,q^l} if l is odd. Hence the statements follow for ψ. $\square$

The following lemma is particularly useful in computing the values of Weil characters at unipotent elements:

Lemma 3.8. *Let $g \in G = Sp_{2n}(q)$ be any element and $i > 0$. If $(|g|, q-1) = 1$, then $\tau_n^i(g) = \rho_n^1(g) + \rho_n^2(g) + 1$. If $(|g|, q+1) = 1$, then $\zeta_n^i(g) = \alpha_n(g) + \beta_n(g) - 1$.*

Proof. Assume $(|g|, q-1) = 1$. Then $\dim \mathrm{Ker}(g - \delta^j) = 0$ if $j > 0$. By (1) it follows for any k that

$$\tau_n^k(g) = -2\delta_{k,0} + \frac{q^{\dim \mathrm{Ker}(g-1)} + \sum_{j=1}^{q-2} \tilde{\delta}^{jk}}{q-1} = -\delta_{k,0} + \frac{q^{\dim \mathrm{Ker}(g-1)} - 1}{q-1}.$$

Using this formula for $k = 0$ and for $k = i > 0$, we get $\tau_n^i(g) - \tau_n^0(g) = 1$, which yields the statement for $\tau_n^i(g)$ because of (3). The proof for $\zeta_n^i(g)$ is similar. □

4. Cross characteristic representations of small groups

In this section we consider the groups $Sp_4(q)$, $Sp_6(2)$, and $Sp_8(2)$.

The character table of $Sp_4(q)$ has been computed in [E], and we will keep the notation used there.

Proposition 4.1. *Let $G = Sp_4(q)$ and suppose that $\varphi \in \mathrm{IBr}_\ell(G)$ satisfies $(\mathcal{W}_2^\varepsilon)$ for some $\varepsilon = \pm$. Then one of the following holds:*

(i) *φ is either trivial or a Weil character.*

(ii) *$q = 2$, $\ell = 3$, $\varepsilon = +$, and φ is the unique irreducible 3-Brauer character of degree 6.*

(iii) *$q = 2$, $\ell \neq 3$, $\varepsilon = +$, and $\varphi = \widehat{\chi_{12}(1)}$ (of degree 10).*

Conversely, if φ satisfies one of (i) – (iii), *then φ satisfies $(\mathcal{W}_2^\varepsilon)$ for some $\varepsilon = \pm$.*

Proof. The Levi subgroup $L_2 = GL_2(q)$ acts on $Z_2^\#$ with three orbits of length $q-1$, resp. q^2-1, $(q^2-1)(q-1)$, whose representatives belong to class A_{31}, resp. A_2, A_{32}, of G. On the other hand, L_2 acts on $\mathrm{IBr}_\ell(Z_2) \setminus \{1\}$ with three orbits, $\mathcal{O}_1$, $\mathcal{O}_2^+$, and $\mathcal{O}_2^-$. Define

$$\omega_1 = \sum_{\alpha \in \mathcal{O}_1} \alpha, \ \omega_2^+ = \sum_{\alpha \in \mathcal{O}_2^+} \alpha, \ \omega_2^- = \sum_{\alpha \in \mathcal{O}_2^-} \alpha.$$

It is easy to compute the character values of these three characters on Q_2:

	1	A_{31}	A_2	A_{32}
ω_1	q^2-1	q^2-1	-1	-1
ω_2^+	$q(q^2-1)/2$	$-q(q+1)/2$	$q(q-1)/2$	$-q/2$
ω_2^-	$q(q-1)^2/2$	$-q(q-1)/2$	$-q(q-1)/2$	$q/2$

Therefore, $\chi|_{Z_2} = a(\chi)\cdot 1 + b(\chi)\omega_1 + c(\chi)\omega_2^+ + d(\chi)\omega_2^-$, where

$$(7) \qquad \begin{cases} c(\chi) = & (\chi(1) + (q-1)\chi(A_2) - \chi(A_{31}) - (q-1)\chi(A_{32}))/q^3, \\ d(\chi) = & (\chi(1) - (q+1)\chi(A_2) - \chi(A_{31}) + (q+1)\chi(A_{32}))/q^3. \end{cases}$$

Clearly, φ satisfies $(\mathcal{W}_2^+)$ if and only if $d(\varphi) = 0$, and φ satisfies $(\mathcal{W}_2^-)$ if and only if $c(\varphi) = 0$. Hence the proposition in the case of complex representations follows from direct calculation using [E]. The irreducible Brauer characters of G are also available from [Wh1], and direct calculation proves the proposition in the modular case. Observe that the character φ mentioned in (ii) is $\widehat{\theta}_4 + 1 - \widehat{\theta}_2 - \widehat{\theta}_3 - \widehat{\theta}_5$.

For the converse, we apply (7) to the character τ_2. If $g = \begin{pmatrix} I_2 & X \\ 0 & I_2 \end{pmatrix}$ is any element in Z_2, then $\tau_2(g) = q^{2+\dim \operatorname{Ker}(X)}$. It follows that τ_2 takes value q^2, resp. q^3, q^2, on classes A_{31}, resp. A_2, A_{32}. Hence

$$\tau_2|_{Z_2} = (2q^2-1)\cdot 1_{Z_2} + (q-1)\omega_1 + (2q-2)\omega_2^+. \tag{8}$$

In particular, $(\mathcal{W}_2^+)$ holds for τ_2. Similarly,

$$\zeta_2|_{Z_2} = 1_{Z_2} + (q+1)\omega_1 + (2q+2)\omega_2^-. \tag{9}$$

In particular, $(\mathcal{W}_2^-)$ holds for ζ_2. Now if φ is trivial or a Weil character, then by definition φ is a constituent of $\widehat{\tau_2}$ or $\widehat{\zeta_2}$, and therefore φ satisfies $(\mathcal{W}_2^\varepsilon)$ for some ε. ☐

Corollary 4.2. *Let $\varphi \in \mathrm{IBr}_\ell(Sp_{2n}(q))$ with $n \geq 2$. If φ is a linear-Weil character, then φ satisfies $(\mathcal{W}_2^+)$. If φ is a unitary-Weil character, then φ satisfies $(\mathcal{W}_2^-)$.*

Proof. Let $G = Sp_{2n}(q)$ and consider $H = Stab_G(e_3, \dots, e_n, f_3, \dots, f_n)$. Then $H \simeq Sp_4(q)$. Moreover, if we consider the parabolic subgroup $P_2(H) := Stab_H(\langle e_1, e_2\rangle_{\mathbb{F}_q})$ of H, then its subgroup $Z_2(H) := O_2(P_2(H))$ is exactly the subgroup Z_2 for the parabolic subgroup $P_2 = Stab_G(\langle e_1, e_2\rangle_{\mathbb{F}_q})$ of G. Observe that $\tau_n|_{Z_2} = q^{2n-4}\cdot\tau_2|_{Z_2}$, and $\zeta_n|_{Z_2} = q^{2n-4}\cdot\zeta_2|_{Z_2}$. Hence $(\mathcal{W}_2^+)$ holds for τ_n by (8), and $(\mathcal{W}_2^-)$ holds for ζ_n by (9). ☐

One of the main goals of the paper is to prove that the converse of Corollary 4.2 is also true (except for $(n,q) = (2,2)$ and $(3,2)$).

Lemma 4.3. *Let $G = Sp_6(2)$ or $Sp_8(2)$, and $\varphi \in \mathrm{IBr}_\ell(G)$. Then φ satisfies $(\mathcal{W}_2^\varepsilon)$ for some $\varepsilon = \pm$ if and only if one of the following holds:*
(i) *φ is either trivial or a Weil character.*
(ii) *$G = Sp_6(2)$, $\varepsilon = +$ and $\varphi = \widehat{\chi_4}$ (where χ_4 is an irreducible complex character of G of degree* 21 *in the notation of* [Atlas]*).*

Proof. Consider a standard subgroup $H \simeq Sp_4(2)$. Then elements of class A_2, resp. A_{31}, A_{32}, of H (in the notation of [E]) belong to class $2A$, resp. $2B$, $2C$, of G (in the notation of [Atlas]). By Corollary 3.5, φ satisfies $(\mathcal{W}_2^+)$ if and only if $(\mathcal{W}_2^+)$ holds for $\varphi|_H$, which is, by the virtue of (7), equivalent to

$$\chi(1) - 3\chi(2A) - \chi(2B) + 3\chi(2C) = 0.$$

Similarly, φ satisfies $(\mathcal{W}_2^-)$ if and only if

$$\chi(1) + \chi(2A) - \chi(2B) - \chi(2C) = 0.$$

Now the lemma follows from direct calculation using character values of G given in [Atlas] and [JLPW]. (Note that in the case $G = Sp_8(2)$, Proposition 5.7 (below) implies that φ can satisfy $(\mathcal{W}_2^\varepsilon)$ only when for any nontrivial Q_1-character λ occurring in $\varphi|_{Q_1}$, the inertia group $Stab_{P_1}(\lambda)$ fixes a subspace of dimension 1 on the corresponding homogeneous component, and hence $\varphi(1) \leq 9180$. Thus it suffices to check the irreducible Brauer characters of degree ≤ 9180.) ☐

5. The local property $(\mathcal{W}_1)$

Now we turn our attention to Q_1. Clearly, Q_1 is elementary abelian. Next, $L_1 = L_1' \times T_1$, where $L_1' \simeq Sp_{2n-2}(q)$ and $T_1 \simeq \mathbb{Z}_{q-1}$. We will write elements of $P_1 = Stab_G(\langle e_1 \rangle_{\mathbb{F}_q})$ with respect to the basis $(e_1, e_2, \ldots, e_n, f_2, \ldots, f_n, f_1)$. The conjugation by $\mathrm{diag}(1, X, 1)$, where $X \in Sp_{2n-2}(q)$, sends any $[A, C] \in Q_1$ to $[XA, C]$, and the conjugation by $\boldsymbol{c}_\alpha := \mathrm{diag}(\alpha, I_{2n-2}, \alpha^{-1}) \in T_1$, where $\alpha \in \mathbb{F}_q^\bullet$, sends $[A, C]$ to $[\alpha A, \alpha^2 C]$. The map $[A, C] \mapsto C$ defines an L_1'-invariant nondegenerate quadratic form on $Q_1 \simeq \mathbb{F}_q^{2n-1}$. We recall the following

Lemma 5.1 ([LaS, Lemma 2.2])**.** *Let V be an n-dimensional vector space over a field $\mathbb{F}_q$, $q = p^f$, and let $\mathbb{F}$ be an algebraically closed field of characteristic other than p. If λ is a nontrivial linear character of (the additive group) V over $\mathbb{F}$, then* $\mathrm{Ker}(\lambda)$ *contains a unique hyperplane of V.* □

Lemma 5.1 implies that P_1 acts on $\mathrm{IBr}_\ell(Q_1)$ with four orbits: $\{1_{Q_1}\}$, $\mathcal{O}_1$ of length $q^{2n-2} - 1$, and $\mathcal{O}_2^\varepsilon$ of length $q^{n-1}(q^{n-1} + \varepsilon)(q-1)/2$ for $\varepsilon = \pm$. Here $\mathcal{O}_1$ is the set of nontrivial characters that are trivial on $Z_1 = Z(P_1')$, and $\mathcal{O}_2^\varepsilon$ consists of nontrivial characters that are trivial on a hyperplane of type ε (with respect to the above quadratic form). Denote $K_\lambda := Stab_{P_1}(\lambda)$, $I_\lambda := Stab_{L_1}(\lambda)$, $J_\lambda := Stab_{L_1'}(\lambda)$. In particular, $I_\lambda = J_\lambda \simeq O_{2n-2}^\varepsilon(q)$ for any $\lambda \in \mathcal{O}_2^\varepsilon$.

Definition 5.2. Let V be any $\mathbb{F}G$-module. We say that V has property $(\mathcal{W}_1)$ if for any nontrivial linear character λ occuring in $V|_{Q_1}$, all I_λ-composition factors on the λ-homogeneous component V_λ of V are of dimension 1.

Even though property $(\mathcal{W}_2^\varepsilon)$ is more transparent than $(\mathcal{W}_1)$, it turns out that it is more convenient to work with $(\mathcal{W}_1)$ while classifying low-dimensional representations. In order to clarify the relationship between $(\mathcal{W}_2^\varepsilon)$ and $(\mathcal{W}_1)$ we need some auxiliary statements.

Lemma 5.3. *Let χ be any Brauer character of a subgroup I of a finite group X. Then for any subgroup Y of X, $\mathrm{Ind}_I^X(\chi)|_Y$ contains the Brauer character $\mathrm{Ind}_{Y \cap I}^Y(\chi|_{Y \cap I})$.*

Proof. By Mackey's formula, $\mathrm{Ind}_I^X(\chi)|_Y = \sum_{t \in Y \backslash X / I} \mathrm{Ind}_{Y \cap I^t}^Y(\chi^t|_{Y \cap I^t})$, where $I^t = tIt^{-1}$ and $\chi^t(x) = \chi(t^{-1}xt)$. Hence the statement follows. □

Lemma 5.4. *Consider the subgroup Z_2 of $H = Sp_4(q)$.*

(i) *Let* $A = \left\{ [a]_1 := \begin{pmatrix} 1 & 0 & a & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid a \in \mathbb{F}_q \right\}$. *If ν is any $\mathbb{F}A$-character which is not trivial on A, then $\mathrm{Ind}_A^{Z_2}(\nu)$ intersects all three orbits $\mathcal{O}_1$, $\mathcal{O}_2^+$, and $\mathcal{O}_2^-$ of Z_2-characters.*

(ii) *Let* $A = \left\{ [a]_2 := \begin{pmatrix} 1 & 0 & 0 & a \\ 0 & 1 & a & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid a \in \mathbb{F}_q \right\}$. *If ν is any $\mathbb{F}A$-character which is not trivial on A, then $\mathrm{Ind}_A^{Z_2}(\nu)$ intersects both orbits $\mathcal{O}_2^+$ and $\mathcal{O}_2^-$ of Z_2-characters.*

(iii) *Let* $A = \left\{ [a,b]_3 := \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & b & b^2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid a = 0,1,\ b \in \mathbb{F}_q \right\}$. *Suppose that* ν *is any* $\mathbb{F}A$*-character whose generalized kernel does not contain* $[1,0]_3$ *and* $[0,1]_3$. *Then* $\mathrm{Ind}_A^{Z_2}(\nu)$ *intersects the orbit* $\mathcal{O}_2^+$ *of* Z_2*-characters. If* $q \geq 4$*, then* $\mathrm{Ind}_A^{Z_2}(\nu)$ *also intersects* $\mathcal{O}_2^-$.

Proof. (i) Since ν is not trivial on A, ν contains a nontrivial A-character $\mu : [a]_1 \mapsto (-1)^{\mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_2}(a\alpha)}$ for some $\alpha \in \mathbb{F}_q^\bullet$. Choose $\beta, \gamma \in \mathbb{F}_q$ such that the polynomial $\alpha t^2 + \gamma t + \beta$ is irreducible over $\mathbb{F}_q$. Consider the Z_2-characters $\lambda_i = \lambda_{B_i}$ for $i = 1,2,3$, where $B_1 = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}$, $B_2 = \begin{pmatrix} \alpha & 1 \\ 0 & 0 \end{pmatrix}$, $B_3 = \begin{pmatrix} \alpha & \gamma \\ 0 & \beta \end{pmatrix}$. Clearly, $\lambda_i|_A = \mu$, whence $\mathrm{Ind}_A^{Z_2}(\nu)$ contains λ_i for $i = 1,2,3$. It remains to observe that $\lambda_1 \in \mathcal{O}_1$, $\lambda_2 \in \mathcal{O}_2^+$, and $\lambda_3 \in \mathcal{O}_2^-$.

(ii) Since ν is not trivial on A, ν contains a nontrivial A-character $\mu : [a]_2 \mapsto (-1)^{\mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_2}(a\gamma)}$ for some $\gamma \in \mathbb{F}_q^\bullet$. Choose $\alpha, \beta \in \mathbb{F}_q$ such that the polynomial $\alpha t^2 + \gamma t + \beta$ is irreducible over $\mathbb{F}_q$. Consider the Z_2-characters $\lambda_i = \lambda_{B_i}$ for $i = 1,2$, where $B_1 = \begin{pmatrix} \alpha & \gamma \\ 0 & 0 \end{pmatrix}$, $B_2 = \begin{pmatrix} \alpha & \gamma \\ 0 & \beta \end{pmatrix}$. Clearly, $\lambda_i|_A = \mu$, whence $\mathrm{Ind}_A^{Z_2}(\nu)$ contains λ_i for $i = 1,2$. It remains to observe that $\lambda_1 \in \mathcal{O}_2^+$ and $\lambda_2 \in \mathcal{O}_2^-$.

(iii) Since the generalizer kernel of ν does not contain $[1,0]_3$, ν contains an A-character $\mu : [a,b]_3 \mapsto (-1)^{\mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_2}(b\gamma)}$ for some $\gamma \in \mathbb{F}_q$. Consider the Z_2-character λ_B, where $B = \begin{pmatrix} 0 & 1 \\ 0 & \gamma^2+1 \end{pmatrix}$. Clearly, $\lambda_B|_A = \mu$, whence $\mathrm{Ind}_A^{Z_2}(\nu)$ contains λ_B, and $\lambda_B \in \mathcal{O}_2^+$.

Next assume that $q \geq 4$. Since the generalized kernel of ν does not contain $[0,1]_3$, ν contains an A-character $\mu' : [a,b]_3 \mapsto (-1)^{a\alpha + \mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_2}(b\beta)}$ for some $\alpha = 0,1$ and $\beta \in \mathbb{F}_q^\bullet$. Since $q \geq 4$, there is an $x \in \mathbb{F}_q^\bullet$ such that $\mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_2}(x) = \alpha$. Let $V_2 = \{t + t^2 \mid t \in \mathbb{F}_q\}$. Since $|V_2| = q/2 \leq q-2$, there is a $y_0 \in \mathbb{F}_q^\bullet \setminus (1 + x^{-1}V_2)$. Let $y = \beta y_0^{-1/2} \in \mathbb{F}_q^\bullet$. Then $\beta^2 = y^2 y_0 \notin y^2 + y^2 x^{-1} V_2$. Finally, let $z = \beta^2 - y^2$ and $C = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$. The choice of x, y, z ensures that $\lambda_C|_A = \mu'$, whence $\mathrm{Ind}_A^{Z_2}(\nu)$ contains λ_C. Claim that $\lambda_C \in \mathcal{O}_2^-$. Assume the contrary. Then there is some $s \in \mathbb{F}_q$ such that $xs^2 + ys + z = 0$. It follows that $xy^{-2}(\beta^2 - y^2) = t^2 + t \in V_2$ for $t = xsy^{-1}$, i.e. $\beta^2 \in y^2 + y^2 x^{-1} V_2$, a contradiction. $\square$

Lemma 5.5. *Consider the subgroup*

$$Z_4 = O_2(Stab_H(\langle e_1, \dots, e_4 \rangle_{\mathbb{F}_q})) = \left\{ [X]_4 := \begin{pmatrix} I_4 & X \\ 0 & I_4 \end{pmatrix} \mid X \in \mathcal{H}_4(q) \right\}$$

of $H = Sp_8(q)$*, and let* $A = Stab_{Z_4}(f_4)$. *Suppose that* ν *is any* $\mathbb{F}A$*-character such that* $\mathrm{Ker}(\nu) \not\geq A_0$*, where* $A_0 = \{[X]_4 \in A \mid X \in \mathcal{H}_4^0(q)\}$. *Then* $\mathrm{Ind}_A^{Z_4}(\nu)$ *intersects at least one of the two orbits* $\mathcal{O}_4^+$ *and* $\mathcal{O}_4^-$ *of* Z_4*-characters.*

Proof. Clearly, any linear character of A is of the form $\mu_C : [X]_4 \mapsto (-1)^{\mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_2}(\mathrm{Tr}(CX))}$ for some $C \in M_4(q)$ with the last column and the last row equal to 0. We may also consider the corresponding quadratic form q_C on $\langle f_1, \dots, f_4 \rangle_{\mathbb{F}_q}$. Observe that if $\mathrm{rank}(q_C) \leq 1$, then C is diagonal and so $\mathrm{Ker}(\mu_C) \geq A_0$. Hence our assumption on ν implies that ν contains a character μ_C with $\mathrm{rank}(q_C) \geq 2$. We may choose f_3 such that f_3 is contained in the radical of the bilinear form associated to q_C. Also, $\mu_C = \mu_{C'}$ whenever $C - C'$ belongs to $\mathcal{H}_4^0(q)$ and has the last column and the last row equal to 0. Hence we may assume that $C = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & e & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ for some $a, b, c, d, e \in \mathbb{F}_q$ and $b \neq c$. Define $B = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & e & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Then $\lambda_B|_A = \mu_C$, whence $\mathrm{Ind}_A^{Z_4}(\nu)$ contains λ_B. It remains to observe that $\mathrm{rank}(q_B) = 4$. $\square$

Lemma 5.6. *Consider the subgroup*

$$Z_3 = O_2(Stab_H(\langle e_1, e_2, e_3 \rangle_{\mathbb{F}_q})) = \left\{ [X]_5 := \begin{pmatrix} I_3 & X \\ 0 & I_3 \end{pmatrix} \mid X \in \mathcal{H}_3(q) \right\}$$

of $H = Sp_6(q)$, and let $A = Stab_{Z_3}(f_3)$. Suppose that ν is any $\mathbb{F}A$-character such that $\mathrm{Ker}(\nu) \not\geq A_0$, where $A_0 = Stab_{Z_3}(f_1, f_3)$. Then $\mathrm{Ind}_A^{Z_3}(\nu)$ intersects the orbit $\mathcal{O}_3$ of Z_3-characters.

Proof. We assume that ν contains the A-character $\mu_C \colon [X]_5 \mapsto (-1)^{\mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_2}(\mathrm{Tr}(CX))}$ for some $C = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 0 \end{pmatrix}$, with $A_0 \not\leq \mathrm{Ker}(\mu_C)$. The last condition means that $d \neq 0$. We may also consider the corresponding quadratic form q_C on $\langle f_1, f_2, f_3 \rangle_{\mathbb{F}_q}$. If $\mathrm{rank}(q_C) = 2$, i.e. $b \neq c$, then define $B := \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$. If $\mathrm{rank}(q_C) = 1$, i.e. $b = c$, then define $B := \begin{pmatrix} a & 0 & 1 \\ 0 & d & 0 \\ 0 & 0 & 0 \end{pmatrix}$. In both cases, $\mathrm{rank}(q_B) = 3$, and $\lambda_B|_A = \mu_C$, whence $\mathrm{Ind}_A^{Z_3}(\nu)$ contains λ_B. $\square$

Proposition 5.7. *Let $G = Sp_{2n}(q)$ with $n \geq 3$ and $(n, q) \neq (3, 2)$. Suppose that an $\mathbb{F}G$-module V satisfies $(\mathcal{W}_2^\varepsilon)$ for some $\varepsilon = \pm$. Then V also satisfies $(\mathcal{W}_1)$.*

Proof. Assume the contrary. Then there is a nontrivial linear character λ of Q_1 such that $I_\lambda := Stab_{L_1}(\lambda)$ has a composition factor of degree > 1 on V_λ, the λ-homogeneous component of V. Clearly, V contains the P_1-submodule $V' := \sum_{\lambda' \in \mathcal{O}} V_{\lambda'}$ and $V' = \mathrm{Ind}_{K_\lambda}^{P_1}(V_\lambda)$, where $\mathcal{O}$ is the orbit containing λ and $K_\lambda = Stab_{P_1}(\lambda)$. Consider the standard subgroup $H = Stab_G(e_3, \dots, e_n, f_3, \dots, f_n) \simeq Sp_4(q)$ of G and the subgroup $Z_2 = Stab_H(e_1, e_2)$ of H. Let K denote the kernel of K_λ on V_λ.

1) First we assume that $\lambda \in \mathcal{O}_1$. We will assume that $P_1' = Stab_G(e_3)$. Then we may identify $K_\lambda \cap P_1'$ with $Stab_G(e_3, f_2 + \langle e_3 \rangle_{\mathbb{F}_q})$. In this case $K_\lambda \cap Z_2 = Stab_{Z_2}(f_2)$ is the subgroup A defined in Lemma 5.4(i). Assume $V_\lambda|_A$ contains a nontrivial linear character ν. By Lemma 5.3, $V'|_{Z_2}$ contains $\mathrm{Ind}_A^{Z_2}(\nu)$ (as $Z_2 < P_1'$), and $\mathrm{Ind}_A^{Z_2}(\nu)$ intersects both $\mathcal{O}_2^+$ and $\mathcal{O}_2^-$ by Lemma 5.4(i). It follows that V cannot satisfy $(\mathcal{W}_2^\varepsilon)$, a contradiction. Therefore A acts trivially on V_λ.

We have shown that $K \geq A$. Observe that

$$K_\lambda \cap P'_1 = Q_1 : J_\lambda,$$

where $J_\lambda := Stab_{L'_1}(\lambda) = Stab_G(e_3, f_3, f_2)$ plays the role of the subgroup P'_1 in the standard subgroup $Stab_G(e_3, f_3) \simeq Sp_{2n-2}(q)$. Next, $J_\lambda = [q^{2n-3}] : S$, where $S = Stab_G(e_2, e_3, f_2, f_3) \simeq Sp_{2n-4}(q)$. Clearly, $A < K \cap S$. Thus $K \cap S$ is a normal subgroup of $S \simeq Sp_{2n-4}(q)$ that contains a long root subgroup A. It follows that $K \geq S$. Now $K \cap J_\lambda$ is a normal subgroup of $J_\lambda = [q^{2n-3}] : Sp_{2n-4}(q)$ that contains $Sp_{2n-4}(q)$. Since $n \geq 3$, this implies that $K \geq J_\lambda$. Observe that Q_1 acts on V_λ as scalars ± 1, and $I_\lambda / J_\lambda \simeq \mathbb{Z}_{q-1}$. It follows that all composition factors of the I_λ-module V_λ are of dimension 1, a contradiction.

2) Next we assume that $\lambda \in \mathcal{O}_2^\alpha$ for some $\alpha = \pm$. We will assume that $P_1 = Stab_G(\langle e_n \rangle_{\mathbb{F}_q})$. In this case $I_\lambda = O(Q) < L'_1 = Stab_G(e_n, f_n)$, where Q is a quadratic form of type α on $\langle e_1, \ldots, e_{n-1}, f_1, \ldots, f_{n-1} \rangle_{\mathbb{F}_q}$ (and the associated bilinear form is $(\cdot, \cdot)$).

Assume $n \geq 4$ or $(n, \alpha) = (3, +)$. Then we may choose e_1, e_2, f_1, f_2 such that Q is totally singular on $\langle e_1, e_2 \rangle_{\mathbb{F}_q}$. In this case, $K_\lambda \cap Z_2$ is the subgroup A defined in Lemma 5.4(ii). Assume $V_\lambda|_A$ contains a nontrivial linear character ν. By Lemma 5.3, $V'|_{Z_2}$ contains $\mathrm{Ind}_A^{Z_2}(\nu)$ (as $Z_2 < P'_1$), and $\mathrm{Ind}_A^{Z_2}(\nu)$ intersects both $\mathcal{O}_2^+$ and $\mathcal{O}_2^-$ by Lemma 5.4(ii). It follows that V cannot satisfy $(\mathcal{W}_2^\varepsilon)$, a contradiction. Therefore A acts trivially on V_λ. We have shown that $K \cap I_\lambda$ contains the subgroup A of order q. Recall that $I_\lambda = O_{2n-2}^\alpha(q) = \Omega_{2n-2}^\alpha(q) : 2$. We claim that $K \geq \Omega_{2n-2}^\alpha(q)$. Indeed, if $n \geq 4$, then $\Omega_{2n-2}^\alpha(q)$ is simple and it is the unique proper nontrivial normal subgroup of I_λ, whence $K \geq \Omega_{2n-2}^\alpha(q)$. Assume $(n, \alpha) = (3, +)$. Since $(n, q) \neq (3, 2)$, we have $q \geq 4$. Now $K \cap \Omega_4^+(q)$ contains the subgroup $A \cap \Omega_4^+(q)$ of order at least 2, and $\Omega_4^+(q) = S_1 \times S_2$ with $S_i \simeq SL_2(q)$, whence $K \cap \Omega_4^+(q)$ is either S_1, S_2, or $S_1 \times S_2$. But any element from $O_4^+(q) \setminus \Omega_4^+(q)$ interchanges S_1 and S_2, hence $K \geq \Omega_4^+(q)$ as stated. Now it is clear that all composition factors of the I_λ-module V_λ are of dimension 1, again a contradiction.

Assume that $(n, \alpha) = (3, -)$. Then we may choose e_1, e_2, f_1, f_2 such that Q has type $-$, resp. $+$, when restricted to $\langle e_1, f_1 \rangle_{\mathbb{F}_q}$, resp. $\langle e_2, f_2 \rangle_{\mathbb{F}_q}$, and moreover $Q(e_1) = 1$ and $Q(e_2) = Q(f_2) = 0$. In this case, $K_\lambda \cap Z_2$ is the subgroup A defined in Lemma 5.4(iii). Let K^* be the generalized kernel of K_λ on V_λ. Assume $K^* \cap A = 1$. By Lemma 5.3, $V'|_{Z_2}$ contains $\mathrm{Ind}_A^{Z_2}(V_\lambda)$ (as $Z_2 < P'_1$), and $\mathrm{Ind}_A^{Z_2}(V_\lambda)$ intersects both $\mathcal{O}_2^+$ and $\mathcal{O}_2^-$ by Lemma 5.4(iii) since $q \geq 4$. It follows that V cannot satisfy $(\mathcal{W}_2^\varepsilon)$, a contradiction. Hence $K^* \cap A \neq 1$. Observe that $SL_2(q^2)$ is the unique proper nontrivial normal subgroup of $I_\lambda = O_4^-(q) \simeq SL_2(q^2) : 2$, whence $K^* \geq SL_2(q^2)$, $K \geq [K^*, K^*] \geq SL_2(q^2)$. It now follows that all composition factors of the I_λ-module V_λ are of dimension 1, again a contradiction. □

Next we show that in general $(\mathcal{W}_1)$ implies $(\mathcal{W}_2^\varepsilon)$ for some $\varepsilon = \pm$.

Lemma 5.8. (i) *Let $g \in Sp(M) = Sp_{2n}(q)$. Then there is a g-invariant quadratic form on M (that is, polarized to the symplectic form $(\cdot, \cdot)$ on M).*

(ii) *Let $\kappa : O_{2n}^\epsilon(q) \to \{\pm 1\}$ be defined by $\kappa(g) = (-1)^{\dim_{\mathbb{F}_q} \mathrm{Ker}(g-1)}$. Then κ is actually a group homomorphism, and* $\mathrm{Ker}(\kappa) = \Omega_{2n}^\epsilon(q)$.

Proof. (i) Let $g = su$, where s is the semisimple part and u is the unipotent part of g. By [SS, Lemma 4.1], there is a u-invariant quadratic form Q on M that is polarized to $(\cdot,\cdot)$. Let $N = |s|$, and let $\tilde{Q}(x) = \sum_{i=0}^{N-1} Q(s^i(x))$ for any $x \in M$. Then $\tilde{Q}$ is g-invariant and polarized to $(\cdot,\cdot)$ (since N is odd).

(ii) Cf. [Atlas, p. xii]. □

The map κ defined in Lemma 5.8(ii) is called the *quasideterminant*.

Lemma 5.9. *Let $G = Sp_{2n}(q)$ with $n \geq 2$, and consider a subgroup $H = O^{\alpha}_{2n}(q)$ of G for some $\alpha = \pm$. Let μ be a linear character of H. If $(n, q, \alpha) = (2, 2, +)$, then assume in addition that $\mu = 1_H$. Then μ is either 1_H or κ. If $\mu = 1_H$, then all G-composition factors of $\mathrm{Ind}_H^G(\mu)$ are linear-Weil modules and satisfy $\mathcal{O}_2^+$. If $\mu = \kappa$, then all G-composition factors of $\mathrm{Ind}_H^G(\mu)$ are unitary-Weil modules and satisfy $\mathcal{O}_2^-$.*

Proof. We fix some notation: $H_\alpha = O^{\alpha}_{2n}(q)$, $K_\alpha = \Omega^{\alpha}_{2n}(q)$.

1) First we consider the case $\mu = 1_H$ and prove the following formula:

$$\mathrm{Ind}_{H_+}^G(1_{H_+}) + \mathrm{Ind}_{H_-}^G(1_{H_-}) = \tau_n. \tag{10}$$

For, the left-hand side of (10) is exactly the permutation character of G on the set Φ of quadratic forms on M that are polarized to $(\cdot,\cdot)$. Let $g \in G$ be any element. By Lemma 5.8(i), g fixes a point $Q \in \Phi$. Now g fixes $Q' \in \Phi$ if and only if g fixes $\sqrt{Q + Q'} \in \mathrm{Hom}_{\mathbb{F}_q}(M, \mathbb{F}_q) = M^*$. So the value of the left-hand side at g is the number of g-fixed points on M^*, and therefore it equals $\tau_n(g)$. By Corollary 4.2, τ_n satisfies $(\mathcal{W}_2^+)$, so we are done with the case $\mu = 1_H$.

2) Now we assume that $\mu_H \neq 1_H$. By assumption, in this case we have $(n, q, \alpha) \neq (2, 2, +)$. It follows that K_α is perfect, so $\mu = \kappa$. By Corollary 4.2, it suffices to prove

$$\mathrm{Ind}_{H_+}^G(\kappa|_{H_+}) + \mathrm{Ind}_{H_-}^G(\kappa|_{H_-}) = \zeta_n. \tag{11}$$

In view of (10), (11) is equivalent to the statement that the virtual character

$$\phi := \tau_n + \zeta_n - \Big(\mathrm{Ind}_{K_+}^G(1_{K_+}) + \mathrm{Ind}_{K_-}^G(1_{K_-})\Big)$$

is 0.

First we show that $\phi(g) \geq 0$ for all $g \in G$. Indeed, assume $d_g := \dim_{\mathbb{F}_q} \mathrm{Ker}(g-1)$ is even. Then

$$\tau_n(g) + \zeta_n(g) = 2q^{d_g} = 2\tau_n(g) = \Big(\mathrm{Ind}_{H_+}^G(2 \cdot 1_{H_+}) + \mathrm{Ind}_{H_-}^G(2 \cdot 1_{H_-})\Big)(g)$$

(because of (10))

$$\geq \Big(\mathrm{Ind}_{K_+}^G(1_{K_+}) + \mathrm{Ind}_{K_-}^G(1_{K_-})\Big)(g)$$

(as $(H_\alpha : K_\alpha) = 2$), whence $\phi(g) \geq 0$. Assume d_g is odd. Then g cannot be contained in any G-conjugate of any $K_\pm$ by Lemma 5.8(ii), so

$$\Big(\mathrm{Ind}_{K_+}^G(1_{K_+}) + \mathrm{Ind}_{K_-}^G(1_{K_-})\Big)|(g) = 0.$$

Also, $\tau_n(g) + \zeta_n(g) = q^{d_g} + (-q)^{d_g} = 0$, whence $\phi(g) = 0$.

Since τ_n, resp. ζ_n, contains 1_G with multiplicity 2, resp. 0, $(\phi, 1_G)_G = 0$. The latter formula, together with the above statement that $\phi(g) \geq 0$ for all $g \in G$, implies that $\phi \equiv 0$. □

Lemma 5.10. *Let $G = Sp_{2n}(q)$ with $n \geq 2$, $(n, q) \neq (2, 2)$, $(3, 2)$. Let μ be a linear character of P'_1. Then $\mu = 1_{P'_1}$, and all G-composition factors of $\mathrm{Ind}_{P'_1}^G(\mu)$ are linear-Weil modules and satisfy $\mathcal{O}_2^+$.*

Proof. Since $n \geq 2$ and $(n, q) \neq (2, 2)$, all P'_1-orbits on $\mathrm{IBr}_\ell(Q_1)$ are of length > 1. Hence $\mathrm{Ker}(\mu) \geq Q_1$ by Clifford's Theorem. Next, our assumption on n, q implies that $P'_1/Q_1 = Sp_{2n-2}(q)$ is perfect. Thus $\mu = 1_{P'_1}$, whence $\mathrm{Ind}_{P'_1}^G(\mu) = \tau_n - 1_G$, and so we are done by Corollary 4.2. □

Recall that $I_\lambda = Stab_{L_1}(\lambda)$ and $J_\lambda = Stab_{L'_1}(\lambda)$ for any $\lambda \in \mathrm{IBr}_\ell(Q_1)$.

Theorem 5.11. *Let $G = Sp_{2n}(q)$ with $n \geq 3$, $(n, q) \neq (3, 2)$, $(4, 2)$, and V a nontrivial irreducible $\mathbb{F}G$-module. Suppose that V satisfies $(\mathcal{W}_1)$. Then exactly one of the following holds:*

(i) *If λ is any nontrivial linear Q_1-character occurring in $V|_{Q_1}$, then $\lambda \in \mathcal{O}_2^+ \cup \mathcal{O}_2^-$, and $I_\lambda = O_{2n-2}^\pm(q)$ acts via the character κ on the homogeneous component V_λ. Furthermore, V satisfies $(\mathcal{W}_2^-)$.*

(ii) *If λ is any nontrivial linear Q_1-character occurring in $V|_{Q_1}$, then all composition factors of the J_λ-module V_λ are trivial. Furthermore, V satisfies $(\mathcal{W}_2^+)$.*

In particular, $(\mathcal{W}_1)$ implies $(\mathcal{W}_2^\varepsilon)$ for some $\varepsilon = \pm$.

Proof. 1) Define $V_1 = \sum_{\lambda \in \mathcal{O}_1} V_\lambda$, $V_2^\varepsilon = \sum_{\lambda \in \mathcal{O}_2^\varepsilon} V_\lambda$, where V_λ is the λ-homogeneous component of $V|_{Q_1}$ for any $\lambda \in \mathrm{IBr}_\ell(Q_1)$. Since L_1 is transitive on each of these orbits $\mathcal{O}_1$, $\mathcal{O}_2^\varepsilon$, the L_1-module V_1, resp. V_2^ε, is $\mathrm{Ind}_{I_\lambda}^{L_1}(V_\lambda)$. Observe that $L_1 = L'_1 \times \mathbb{Z}_{q-1}$. Moreover, L'_1 acts transitively on $\mathcal{O}_1$, meanwhile each $\mathcal{O}_2^\varepsilon$ splits into $q-1$ L'_1-orbits. It follows by Mackey's formula that $V_1 = \mathrm{Ind}_{J_\lambda}^{L'_1}(V_\lambda)$ and $V_2^\varepsilon = (q-1)\mathrm{Ind}_{J_\lambda}^{L'_1}(V_\lambda)$ as L'_1-modules.

By assumption, all J_λ-composition factors on V_λ are of dimension 1. Consider any $\lambda \in \mathcal{O}_2^\varepsilon$. Then Lemma 5.9 implies that the Brauer character of $J_\lambda = O_{2n-2}^\varepsilon(q)$ on V_λ is $a_\varepsilon \cdot 1_{J_\lambda} + b_\varepsilon \cdot \kappa$ for some $a_\varepsilon, b_\varepsilon \in \mathbb{Z}$, and all L'_1-composition factors of V_2^ε are (linear or unitary) Weil modules. Next let $\lambda \in \mathcal{O}_1$. Then Lemma 5.10 implies that the Brauer character of J_λ on V_λ is $c \cdot 1_{J_\lambda}$ for some $c \in \mathbb{Z}$, and all L'_1-composition factors of V_1 are linear-Weil modules.

Observe that $V = C_V(Q_1) \oplus V_1 \oplus V_2^+ \oplus V_2^-$ as a P_1-module. By Lemma 2.1, all L'_1-composition factors of $C_V(Q_1)$ are also Weil modules.

2) Let $\mathfrak{W}_+$, resp. $\mathfrak{W}_-$, be the set of all linear-Weil, resp. unitary-Weil, irreducible Brauer characters of L'_1 together with the trivial character. Observe that if $\rho \in \mathfrak{W}_+ \cap \mathfrak{W}_-$, then ρ is trivial. Indeed, ρ satisfies both $(\mathcal{W}_2^+)$ and $(\mathcal{W}_2^-)$ by Corollary 4.2. Hence, if Z_2 is a standard subgroup of type Z_2 for L'_1, then all Z_2-characters in $\rho|_{Z_2}$ are contained in the orbit $\mathcal{O}_1$. It follows that $\mathrm{Ker}(\rho)$ contains the subgroup $\{[0, C] \in Z_2 \mid C \in \mathcal{H}_2^0(q)\}$ of order q, and so ρ is trivial by irreducibility. Notice that we have used the assumption $n \geq 3$ here.

Given any $\mathbb{F}L'_1$-module M and any family $\mathcal{X}$ of simple L'_1-modules, there is a largest submodule $M(\mathcal{X})$ of M with all composition factors belonging to $\mathcal{X}$; cf. [GMST, Lemma 4.3].

3) Let $\lambda \in \mathcal{O}_2^\varepsilon$. Then $I_\lambda = J_\lambda = O_{2n-2}^\varepsilon(q)$, and the perfect subgroup $\Omega_{2n-2}^\varepsilon(q)$ of J_λ acts trivially on V_λ. Also, Q_1 acts scalarly on V_λ (via the character λ). Hence, the K_λ-module V_λ is semisimple, where $K_\lambda := Stab_{P_1}(\lambda)$. In particular, $V_\lambda = V_\lambda^+ \oplus V_\lambda^-$, where V_λ^+, resp. V_λ^-, affords the J_λ-character $a_\varepsilon \cdot 1_{J_\lambda}$, resp. $b_\varepsilon \cdot \kappa$.

Let $T = [V, Q_1]$, $T_+ = V_1 \oplus \mathrm{Ind}_{K_{\lambda_1}}^{P_1}(V_{\lambda_1}^+) \oplus \mathrm{Ind}_{K_{\lambda_2}}^{P_1}(V_{\lambda_2}^+)$, $T_- = \mathrm{Ind}_{K_{\lambda_1}}^{P_1}(V_{\lambda_1}^-) \oplus \mathrm{Ind}_{K_{\lambda_2}}^{P_1}(V_{\lambda_2}^-)$, for fixed $\lambda_1 \in \mathcal{O}_2^+$ and $\lambda_2 \in \mathcal{O}_2^-$. Then $T = T_+ \oplus T_-$ as P_1-modules.

Claim that $T(\mathfrak{W}_+) = T_+$. Indeed, $T(\mathfrak{W}_+) \supseteq T_+$ by Lemmas 5.9 and 5.10. Assume that $T(\mathfrak{W}_+) \neq T_+$. Then $T_- \simeq T/T_+$ contains a simple L_1'-submodule with character $\rho \in \mathfrak{W}_+$. On the other hand, the construction of T_- implies by Lemma 5.9 that $\rho \in \mathfrak{W}_-$. Hence $\rho = 1_{L_1'}$ according to 2). Thus T_- contains the trivial L_1'-module as a submodule. We may therefore assume that

$$0 \neq \mathrm{Hom}_{L_1'}\left(1_{L_1'}, \mathrm{Ind}_{J_{\lambda_i}}^{L_1'}(\kappa)\right) \simeq \mathrm{Hom}_{J_{\lambda_i}}(1_{J_{\lambda_i}}, \kappa)$$

for some $i = 1$ or 2. But the last hom-space is 0 as $J_{\lambda_i} = O_{2n-2}^{\pm}(q)$ and $\kappa = -1$ on $O_{2n-2}^{\pm}(q) \setminus \Omega_{2n-2}^{\pm}(q)$, a contradiction.

Thus $T(\mathfrak{W}_+) = T_+$; in particular, $T(\mathfrak{W}_+)$ is P_1-stable.

4) Let $U = C_V(Q_1)$. Since Q_1 acts trivially on U and $L_1 = L_1' \times \mathbb{Z}_{q-1}$, $U(\mathfrak{W}_+)$ is also P_1-stable. It follows that $V(\mathfrak{W}_+) = T(\mathfrak{W}_+) \oplus U(\mathfrak{W}_+)$ is P_1-stable. On the other hand, it is clear that $V(\mathfrak{W}_+)$ is stable under a standard subgroup $S := Sp_2(q)$ that centralizes L_1'. Consequently, $V(\mathfrak{W}_+)$ is stable under $\langle P_1, S\rangle = G$, and so $V(\mathfrak{W}_+) = 0$ or V by irreducibility.

Assume $V(\mathfrak{W}_+) = 0$. Then $V_1 = 0$ and $a_\varepsilon = 0$ for all $\varepsilon = \pm$. Also, $T = T_-$, and so all L_1'-composition factors of T are unitary-Weil modules. The same holds for U as well by Lemma 2.1. Hence the L_1'-module V satisfies $(\mathcal{W}_2^-)$ by Corollary 4.2, and (i) holds.

Assume $V(\mathfrak{W}_+) = V$. Then $T_- = 0$, and $b_\varepsilon = 0$ for all $\varepsilon = \pm$. Also, all L_1'-composition factors of V are linear-Weil modules. Hence the L_1'-module V satisfies $(\mathcal{W}_2^+)$ by Corollary 4.2, and (ii) holds.

Finally, the argument in 2) (applied to G in place of L_1') shows that exactly one of the cases (i), (ii) holds for V, since V is nontrivial. $\square$

Let $b^\epsilon(n, q)$ denote the improved Landazuri-Seitz-Zalesskii lower bound for faithful $\mathbb{F}\Omega_{2n}^\epsilon(q)$-representations as stated in [Ho]. In particular,

$$b^-(n, q) = \frac{(q^n + 1)(q^{n-1} - q)}{q^2 - 1} - 1$$

if $n \geq 4$, except for $(n, q) = (4, 2)$, $(4, 4)$, $(5, 2)$, where one has to decrease the bound by 2. Define

$$\mathfrak{d}(n, q) = \begin{cases} b^-(n-1, q) \cdot q^{n-1}(q^{n-1} - 1)(q-1)/2, & \text{if } n \geq 5, \\ (q^4 - 1)(q^3 - 1)q^2, & \text{if } n = 4. \end{cases}$$

Observe that

$$\mathfrak{d}(n, q) = \frac{q^{4n-6}}{2} \cdot \left(1 - \frac{1}{q} + O(\frac{1}{q^2})\right)$$

provided that $n \geq 5$.

Theorem 5.12. *Let $G = Sp_{2n}(q)$ with $n \geq 4$ and $(n, q) \neq (4, 2)$, $(5, 2)$, and let $V \in \mathrm{IBr}_\ell(G)$. Then either $\dim(V) \geq \mathfrak{d}(n, q)$, or V satisfies $(\mathcal{W}_1)$.*

Proof. Assume the contrary: $\dim(V) < \mathfrak{d}(n, q)$, but V violates $(\mathcal{W}_1)$. Then there is a nontrivial linear character λ of Q_1 such that I_λ has a composition factor of dimension > 1 on the homogeneous component V_λ of V.

First assume that $\lambda \in \mathcal{O}_2^\varepsilon$. Then $I_\lambda = \Omega^\varepsilon_{2n-2}(q) : 2$ and $\Omega^\varepsilon_{2n-2}(q)$ is simple since $n \geq 4$. The assumption on λ implies that $\dim(V) \geq |\mathcal{O}_2^\varepsilon| \cdot b^\varepsilon(n-1,q) \geq \mathfrak{d}(n,q)$, a contradiction. Hence $\lambda \in \mathcal{O}_1$.

Define $t = 4$ for $n \geq 5$ and $t = 3$ for $n = 4$. Consider the standard subgroup

$$H = Stab_G(e_{t+1}, \dots, e_n, f_{t+1}, \dots, f_n) \simeq Sp_{2t}(q)$$

of G and the subgroup $Z_t = Stab_H(e_1, \dots, e_t)$ of H. Let K denote the kernel of $K_\lambda := Stab_{P_1}(\lambda)$ on V_λ. We will assume that $P_1' = Stab_G(e_{t+1})$. Then we may identify $K_\lambda \cap P_1'$ with $Stab_G(e_{t+1}, f_t + \langle e_{t+1} \rangle_{\mathbb{F}_q})$. In this case $K_\lambda \cap Z_t = Stab_{Z_t}(f_t)$ is the subgroup A defined in Lemma 5.5 when $n \geq 5$ and in Lemma 5.6 when $n = 4$. Assume the subgroup A_0 of A (as defined in Lemma 5.5, respectively in Lemma 5.6) does not act trivially on V_λ. By Lemma 5.3, $V|_{Z_t}$ contains $\mathrm{Ind}_A^{Z_t}(V_\lambda)$ (as $Z_t < P_1'$), and $\mathrm{Ind}_A^{Z_t}(V_\lambda)$ intersects some orbit $\mathcal{O}_t^\epsilon$ of Z_t-characters, according to Lemma 5.5, respectively Lemma 5.6. In other words, the $\mathbb{F}P_t$-module V does not satisfy $(\mathcal{W}_{t-1})$. By Lemma 3.4(ii), $V|_{Z_n}$ does not satisfy $(\mathcal{W}_{t-1})$. In other words, $V|_{Z_n}$ affords some orbit $\mathcal{O}_s^\epsilon$ of Z_n-characters, with $s \geq t$. Observe that

$$|\mathcal{O}_1^\pm| \leq |\mathcal{O}_2^\pm| \leq |\mathcal{O}_3^\pm| \leq \ldots \leq |\mathcal{O}_n^\pm|$$

if $q \geq 4$ or if n is odd, and

$$|\mathcal{O}_1^\pm| \leq |\mathcal{O}_2^\pm| \leq \ldots \leq |\mathcal{O}_{n-2}^\pm| \leq \min\{|\mathcal{O}_{n-1}^\pm|, |\mathcal{O}_n^\pm|\}$$

if $q = 2$ and n is even. Since we are assuming $(n,q) \neq (4,2)$, it follows that $\dim(V) \geq |\mathcal{O}_4^\pm| > \mathfrak{d}(n,q)$ if $n \geq 5$, and $\dim(V) \geq |\mathcal{O}_3| = \mathfrak{d}(n,q)$ if $n = 4$, again a contradiction.

We have shown that $K \geq A_0$. Observe that $K_\lambda \cap P_1' = Q_1 : J_\lambda$, where $J_\lambda := Stab_{L_1'}(\lambda) = Stab_G(e_{t+1}, f_{t+1}, f_t)$ plays the role of the subgroup P_1' in the standard subgroup $Stab_G(e_{t+1}, f_{t+1}) \simeq Sp_{2n-2}(q)$. Next, $J_\lambda = [q^{2n-3}] : S$, where $S = Stab_G(e_t, e_{t+1}, f_t, f_{t+1}) \simeq Sp_{2n-4}(q)$. Clearly, $A_0 < K \cap S$. Thus $K \cap S$ is a normal subgroup of $S \simeq Sp_{2n-4}(q)$ of order $\geq |A_0| \geq q$. Since $n \geq 4$ and $(n,q) \neq (4,2)$, it follows that $K \geq S$. Now $K \cap J_\lambda$ is a normal subgroup of $J_\lambda = [q^{2n-3}] : Sp_{2n-4}(q)$ that contains $Sp_{2n-4}(q)$. Since $n \geq 4$, this implies that $K \geq J_\lambda$. Observe that Q_1 acts on V_λ as scalars ± 1, and $I_\lambda / J_\lambda \simeq \mathbb{Z}_{q-1}$. It follows that all composition factors of the I_λ-module V_λ are of dimension 1, contrary to the choice of λ. $\square$

Remark 5.13. Theorem 5.12 remains true for $(n,q) = (5,2)$ if we set $\mathfrak{d}(5,2) = |\mathcal{O}_2^+| \cdot b^+(4,2) = 3808$.

6. Low-dimensional complex representations of $Sp_{2n}(q)$

Irreducible complex representations of $G = Sp_{2n}(q)$ have been classified up to degree $(q^{2n}-1)/2(q+1)$; cf. [TZ1, Theorem 5.5]. In this section we extend this classification up to degree $(q^{2n}-1)(q^{n-1}-1)(q^{n-1}-q^2)/2(q^4-1)$ (generically). Since the argument is pretty much the same as in [TZ1], we will only sketch the proof. Since q is even, we may identify G with its dual group. *Lusztig's classification of irreducible characters of* G [L] parametrizes $\chi \in \mathrm{Irr}(G)$ by a pair $((s), \chi_u)$, where (s) is a G-conjugacy class of a semisimple element $s \in G$ and χ_u is a unipotent character of $C_G(s)$. In this case $\chi(1) = (G : C_G(s))_{2'} \cdot \chi_u(1)$. Furthermore, unipotent characters of G are parametrized by symbols $\begin{pmatrix} \lambda \\ \mu \end{pmatrix}$ of a certain kind as explained in [C, §13.8].

Theorem 6.1. *Let $G = Sp_{2n}(q)$, $n \geq 1$, and let $\chi \in \mathrm{Irr}(G)$.*

(A) *Suppose $n \geq 3$ and χ is unipotent. Then one of the following holds:*

(A1) $\chi \in \{1_G, \alpha_n, \beta_n, \rho^1_n, \rho^2_n\}$.

(A2) *$n = 3$, $q = 2$, and $\chi(1)$ is at least $q^4(q-1)^2(q^2+q+1)/2$, which is the degree of the unipotent character labelled by* $\begin{pmatrix} 0\ 1\ 2\ 3 \\ 1 \end{pmatrix}$.

(A3) *$n = 3$, $q > 2$, and $\chi(1)$ is at least $q^2(q^4+q^2+1)$, which is the degree of the unipotent character labelled by* $\begin{pmatrix} 0\ 2 \\ n-1 \end{pmatrix}$.

(A4) *$n \geq 4$, and $\chi(1)$ is at least $(q^{2n}-1)(q^{n-1}-1)(q^{n-1}-q^2)/2(q^4-1)$, which is the degree of the unipotent character labelled by* $\begin{pmatrix} 0\ 2\ n-1 \\ - \end{pmatrix}$.

(B) *Suppose χ is not unipotent. Then $\chi(1) \geq (q^{2n}-1)/(q+1)$. Moreover, if $n \geq 3$ and $(n,q) \neq (3,2)$, then one of the following holds:*

(B1) $\chi(1) \geq (q^{2n}-1)(q^{2n-2}-1)/(q^2-1)(q+1)$.

(B2) $\chi \in \{\tau^i_n \mid 1 \leq i \leq q/2-1\} \cup \{\zeta^i_n \mid 1 \leq i \leq q/2\}$.

(B3) *$n = 3, 4$, $\chi(1) = \prod_{i=1}^{n}(q^i + \gamma^i)$, $\gamma = \pm$, and χ is parametrized by $((s), 1_C)$, where $C := C_G(s) \simeq GL^{\gamma}_n(q)$.*

(B4) *$n = 5$, $\chi(1) = \prod_{i=1}^{5}(q^i + (-1)^i)$, and χ is parametrized by $((s), 1_C)$, where $C := C_G(s) \simeq GU_5(q)$.*

(B5) *$n \geq 4$, $q = 2$, $\chi(1) = (q^{2n}-1)(q^{n-1}-\gamma)(q^{n-1}-\gamma q)/2(q^2-1)(q+1)$, $\gamma = \pm$, and χ is parametrized by $((s), \chi_u)$, where $C_G(s) \simeq \mathbb{Z}_{q+1} \times Sp_{2n-2}(q)$ and $\chi_u = 1 \otimes \alpha_{n-1}$, resp. $1 \otimes \beta_{n-1}$, for $\gamma = +$, resp. for $\gamma = -$.*

Proof. (A) Assume that χ is unipotent. Define

$$D(n) = (q^{2n}-1)(q^{n-1}-1)(q^{n-1}-q^2)/2(q^4-1)$$

when $n \geq 4$ and $D(3) = q^2(q^4+q^2+1)$. Notice that $D(n)/D(n-1) < q^5$ and $D(n) < q^{4n-5}/2$, provided that $n \geq 4$. Also, $D(n) > q^{4n-6}/2$ if $n \geq 8$ and $D(n) > q^{4n-7}/2$ if $n \geq 5$.

1) First we observe that the unipotent characters labelled by $\begin{pmatrix} n \\ - \end{pmatrix}$, resp. $\begin{pmatrix} 0\ 1\ n \\ - \end{pmatrix}$, $\begin{pmatrix} 0\ 1 \\ n \end{pmatrix}$, $\begin{pmatrix} 1\ n \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0\ n \\ 1 \end{pmatrix}$, have degree 1, resp. $\alpha_n(1)$, $\beta_n(1)$, $\rho^1_n(1)$, $\rho^2_n(1)$. Let $\mathcal{L}_n$ be the set of these five symbols. We will actually prove by *induction on $n \geq 3$ and $(n,q) \neq (3,2)$* that if $\chi = \chi^{\lambda,\mu}$ with $\begin{pmatrix} \lambda \\ \mu \end{pmatrix} \notin \mathcal{L}_n$, then $\chi(1) \geq D(n)$. The induction base $n = 3$ can be checked using [Lu], and the case $(n,q) = (4,2)$ can be checked using [Atlas]. So for the induction step we will assume that $n \geq 4$ and that $(n,q) \neq (4,2)$. Assume that χ is the unipotent character $\chi^{\lambda,\mu}$ labelled by $\begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} \lambda_1\ \lambda_2\ \lambda_3\ \ldots\ \lambda_a \\ \mu_1\ \mu_2\ \ldots\ \mu_b \end{pmatrix}$, where $0 \leq \lambda_1 < \lambda_2 < \ldots < \lambda_a$, $0 \leq \mu_1 < \mu_2 < \ldots < \mu_b$, $a-b$ is odd and positive, $(\lambda_1, \mu_1) \neq (0,0)$, and

$$\sum_i \lambda_i + \sum_j \mu_j - \left[\left(\frac{a+b-1}{2}\right)^2\right] = n.$$

The integer n is called the rank of $\binom{\lambda}{\mu}$. We will assume $\binom{\lambda}{\mu} \notin \mathcal{L}_n$ and proceed by *induction on* b.

2) Here we consider the case $b = 0$. Then $a \geq 1$ is odd. We may assume that $a \geq 3$, and $\lambda \neq (0, 1, n)$, $(0, 2, n-1)$.

First we assume that $a = 3$. If $\lambda_1 = 0$, then $\lambda = (0, k, n+1-k)$, where $3 \leq k < (n+1)/2$; in particular $n \geq 6$. It is easy to check that $\chi(1) > D(n)$ in this case. If $\lambda_1 \geq 1$, then $\chi(1) > q^{4n-4}/2 > D(n)$, cf. [TZ1, p. 2119]. Now we may assume that $a \geq 5$.

Suppose $\lambda_1 \geq 1$. Consider the unipotent character χ' labelled by the symbol $\binom{\lambda'}{\mu'}$ of rank $n-1$, where $\lambda' = (\lambda_1 - 1, \lambda_2, \ldots, \lambda_a)$, and $\mu' = \mu$. Observe that $\chi(1)/\chi'(1) \geq q^{2(n-\lambda_1)}/2$. But $n - \lambda_1 \geq (a-1)\lambda_1 + (a^2-1)/4 \geq 10$, and $\chi'(1) \geq D(n-1)$ by induction hypothesis. It follows that $\chi(1) > q^{19}D(n-1) > D(n)$. Now we may assume that $\lambda_1 = 0$.

Next we assume that $\lambda_i - \lambda_{i-1} \geq 2$ for some $i \geq 2$. Consider the unipotent character χ' labelled by the symbol $\binom{\lambda'}{\mu'}$ of rank $n-1$, where $\lambda' = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \ldots, \lambda_a)$, and $\mu' = \mu$. Observe that $\chi(1)/\chi'(1) > q^{2(n-\lambda_i)}/2$, and $n-1 \geq 6$. By induction hypothesis, $\chi'(1) \geq D(n-1)$. Hence, if $n - \lambda_i \geq 3$, then $\chi(1) > q^5 D(n-1) > D(n)$. If $n - \lambda_i \leq 2$, then actually $i = a = 5$, $\lambda = (0, 1, 2, 3, n-2)$ and so $\chi(1) > D(n)$.

Now we may assume that $\lambda = (0, 1, 2, \ldots, a-2, a-1)$. Consider $\chi' = \chi^{\lambda', \mu'}$, where $\lambda' = (0, 1, 2, \ldots, a-4, a-3)$, and $\mu' = \mu$. Then $\chi(1)/\chi'(1) \geq q^{4n-2}$, and so $\chi(1) > q^{4n-2} > D(n)$.

3) From now on we may assume that $b \geq 1$. At this point we suppose that $(\lambda_1, \mu_1) \neq (1, 0)$ and $\lambda_1 \geq 1$. Consider the character χ' labelled by the symbol $\binom{\lambda'}{\mu'}$ of rank $n-1$, where $\lambda' = (\lambda_1 - 1, \lambda_2, \ldots, \lambda_a)$ and $\mu' = \mu$. If $n - \lambda_1 \leq 2$, then $n = 4$, $\binom{\lambda}{\mu} = \binom{2\ 3}{0}$ and so $\chi(1) > D(n)$. Assume $n - \lambda_1 \geq 3$. Then $\chi(1)/\chi'(1) > q^{2(n-\lambda_1)}/2 \geq q^5$. In the case $\binom{\lambda'}{\mu'} \in \mathcal{L}_{n-1}$ one can easily check that $\chi(1) > D(n)$. If $\binom{\lambda'}{\mu'} \notin \mathcal{L}_{n-1}$, then $\chi'(1) > D(n-1)$ by induction hypothesis, whence $\chi(1) > D(n)$.

Similarly, for $(\lambda_1, \mu_1) \neq (0, 1)$ and $\mu_1 \geq 1$, we consider the character χ' labelled by the symbol $\binom{\lambda'}{\mu'}$ of rank $n-1$, where $\lambda' = \lambda$ and $\mu' = (\mu_1 - 1, \mu_2, \ldots, \mu_b)$. If $n - \mu_1 \leq 2$, then one directly checks that $\chi(1) > D(n)$. Assume $n - \mu_1 \geq 3$. Then $\chi(1)/\chi'(1) > q^{2(n-\mu_1)}/2 \geq q^5$. In the case $\binom{\lambda'}{\mu'} \in \mathcal{L}_{n-1}$, $\chi(1) > D(n)$ also holds. If $\binom{\lambda'}{\mu'} \notin \mathcal{L}_{n-1}$, then $\chi'(1) > D(n-1)$ by induction hypothesis, whence $\chi(1) > D(n)$.

It remains to consider the case where $\{\lambda_1, \mu_1\} = \{0, 1\}$.

4) Here we suppose that $(\lambda_1, \mu_1) = (1, 0)$ but $\lambda \neq (1, 2, \ldots, a)$. Then there exists an $i \geq 2$ such that $\lambda_i \geq \lambda_{i-1} + 2$. If $a = 2$, then $\binom{\lambda}{\mu} \in \mathcal{L}_n$, hence $a \geq 3$. Consider the character χ' labelled by the symbol $\binom{\lambda'}{\mu'}$ of rank $n-1$, where $\lambda' = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \ldots, \lambda_a)$, and $\mu' = \mu$. Since $a \geq 3$, $\binom{\lambda'}{\mu'} \notin \mathcal{L}_{n-1}$ and so $\chi'(1) > D(n-1)$. Observe that $\chi(1)/\chi'(1) > q^{2(n-\lambda_i+1)}/2$; cf. [TZ1, p. 2121]. Therefore, if $n - \lambda_i \geq 2$, then $\chi(1)/\chi'(1) > q^5$ and we are done. Assume $n - \lambda_i \leq 1$. Then $i = a \geq 3$. If $n - \lambda_a = 1$, then for $n \geq 5$,

$$\begin{aligned}\frac{\chi(1)}{\chi'(1)} &= \frac{q^{2n}-1}{q^{2\lambda_a}-1} \cdot \prod_{i'<a} \frac{q^{\lambda_a} - q^{\lambda_{i'}}}{q^{\lambda_a - 1} - q^{\lambda_{i'}}} \cdot \prod_j \frac{q^{\lambda_a} + q^{\mu_j}}{q^{\lambda_a - 1} + q^{\mu_j}} \\ &\geq \frac{q^{2n}-1}{q^{2n-2}-1} \cdot \frac{q^{n-1}-q}{q^{n-2}-q} \cdot \frac{q^{n-1}-q^2}{q^{n-2}-q^2} \cdot \frac{q^{n-1}+1}{q^{n-2}+1},\end{aligned}$$

and $\chi'(1) > D(n-1)$, whence $\chi(1) > D(n)$. Assume that $n \leq \lambda_a$. Then $\binom{\lambda}{\mu} = \begin{pmatrix} 1, & 2, & \ldots, & a-1, & n \\ & 0, & 1, & \ldots, & a-2 \end{pmatrix}$ with $n = \lambda_a \geq a + 1 \geq 4$. If $a = 3$, then $\binom{\lambda}{\mu} = \begin{pmatrix} 1 & 2 & n \\ & 0 & 1 \end{pmatrix}$, and $\chi(1) > D(n)$. Assume $a \geq 4$. In this case

$$\begin{aligned}\chi(1)/\chi'(1) &= \prod_{j=1}^{a-1} \left(\frac{q^n - q^j}{q^{n-1} - q^j} \cdot \frac{q^n + q^{j-1}}{q^{n-1} + q^{j-1}} \right) \\ &= \frac{(q^n+1)(q^{n-1}-1)}{(q^{n-a}-1)(q^{n-a+1}+1)} > q^{2a-2} \geq q^6,\end{aligned}$$

and we are again done.

Similarly, suppose that $(\lambda_1, \mu_1) = (0, 1)$ but $\mu \neq (1, 2, \ldots, b)$; in particular, $b \geq 2$. Then there exists an index $j \geq 2$ such that $\mu_j \geq \mu_{j-1} + 2$. Consider the character χ' labelled by the symbol $\binom{\lambda'}{\mu'}$ of rank $n-1$, where $\lambda' = \lambda$ and $\mu' = (\mu_1, \ldots, \mu_{j-1}, \mu_j - 1, \mu_{j+1}, \ldots, \mu_b)$. Since $b \geq 2$, $\binom{\lambda'}{\mu'} \notin \mathcal{L}_{n-1}$ and so $\chi'(1) > D(n-1)$. Observe that $\chi(1)/\chi'(1) > q^{2(n-\mu_j+1)}/2$; cf. [TZ1, p. 2122]. Therefore, if $n - \mu_j \geq 2$, then $\chi(1)/\chi'(1) > q^5$ and we are done. Assume $n - \mu_j \leq 1$. Then $j = b$. If $n - \mu_b = 1$, then $(\lambda_1, \lambda_2) = (0, 1)$,

$$\begin{aligned}\frac{\chi(1)}{\chi'(1)} &= \frac{q^{2n}-1}{q^{2\mu_b}-1} \cdot \prod_i \frac{q^{\lambda_i} + q^{\mu_b}}{q^{\lambda_i} + q^{\mu_b - 1}} \cdot \prod_{j'<b} \frac{q^{\mu_b} - q^{\mu_{j'}}}{q^{\mu_b - 1} - q^{\mu_{j'}}} \\ &\geq \frac{q^{2n}-1}{q^{2n-2}-1} \cdot \frac{q^{n-1}+1}{q^{n-2}+1} \cdot \frac{q^{n-1}+q}{q^{n-2}+q} \cdot \frac{q^{n-1}-q}{q^{n-2}-q},\end{aligned}$$

and $\chi'(1) > D(n-1)$, whence $\chi(1) > D(n)$. Assume that $n \leq \mu_b$. Then $\binom{\lambda}{\mu} = \begin{pmatrix} 0, & 1, & 2 & \ldots, & a-2, & a-1 \\ & 1, & 2, & \ldots, & a-2, & n \end{pmatrix}$ with $n = \mu_b \geq a$ and $b = a - 1$. If $a = 3$, then

$\binom{\lambda}{\mu} = \binom{0\ 1\ 2}{1\ n}$, and $\chi(1) > D(n)$. Assume $a \geq 4$. In this case

$$\chi(1)/\chi'(1) = \prod_{i=0}^{a-1} \frac{q^n+q^i}{q^{n-1}+q^i} \cdot \prod_{i=1}^{a-2} \frac{q^n-q^i}{q^{n-1}-q^i} = \frac{(q^n+1)(q^{n-1}-1)}{(q^{n-a}+1)(q^{n-a+1}-1)} > q^{2a-3} \geq q^5,$$

and we are again done.

5) Here we suppose that $\mu_1 = 0$ and $\lambda = (1, 2, \ldots, a)$. Then consider the character χ' labelled by the symbol

$$\binom{\lambda'}{\mu'} = \begin{pmatrix} 0\ 1\ 2\ \ldots\ a \\ \mu_2\ \mu_3\ \ldots\ \mu_b \end{pmatrix}$$

of the same rank n, but with the parameter $b' = b - 1$. If $\binom{\lambda'}{\mu'} \in \mathcal{L}_n$, then $\binom{\lambda}{\mu} = \binom{1\ n}{0} \in \mathcal{L}_n$, and so we are done. Assume $\binom{\lambda'}{\mu'} \notin \mathcal{L}_n$. Then $\chi'(1) > D(n)$ by the induction hypothesis (on b). But $\chi(1) > \chi'(1)$ (cf. [TZ1, p. 2122]), so we are done again.

Similarly, suppose that $\lambda_1 = 0$, $\mu = (1, 2, \ldots, b)$, but $\lambda \neq (0, 1, \ldots, a-1)$. Then there exists an $i \geq 2$ such that $\lambda_i \geq \lambda_{i-1} + 2$. Consider the character χ' labelled by the symbol $\binom{\lambda'}{\mu'}$ of rank $n-1$, where $\lambda' = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \ldots, \lambda_a)$, and $\mu' = \mu$. If $\binom{\lambda'}{\mu'} \in \mathcal{L}_{n-1}$, then $\binom{\lambda}{\mu} = \binom{0\ n}{1} \in \mathcal{L}_n$, and so we are done. So we may assume that $\chi'(1) > D(n-1)$. Observe that $\chi(1)/\chi'(1) > q^{2(n-\lambda_i)+1}/2$; cf. [TZ1, p. 2123]. Therefore, if $n - \lambda_i \geq 3$, then $\chi(1)/\chi'(1) > q^6$ and we are done. Assume $n - \lambda_i \leq 2$. If $i \leq a-1$, then the last condition implies that $i = a-1$ and $(\lambda_{a-1}, \lambda_a) = (n-2, n-1)$, whence

$$\begin{aligned} \frac{\chi(1)}{\chi'(1)} &= \frac{q^{2n}-1}{q^{2\lambda_i}-1} \cdot \frac{q^{\lambda_a}-q^{\lambda_i}}{q^{\lambda_a}-q^{\lambda_i-1}} \cdot \prod_{i'<i} \frac{q^{\lambda_i}-q^{\lambda_{i'}}}{q^{\lambda_i-1}-q^{\lambda_{i'}}} \cdot \prod_j \frac{q^{\lambda_i}+q^{\mu_j}}{q^{\lambda_i-1}+q^{\mu_j}} \\ &\geq \frac{q^{2n}-1}{q^{2n-4}-1} \cdot \frac{q^{n-1}-q^{n-2}}{q^{n-1}-q^{n-3}} \cdot \frac{q^{n-2}-1}{q^{n-3}-1} \cdot \frac{q^{n-2}+q}{q^{n-3}+q} > q^5, \end{aligned}$$

and so we are done. Now we may assume that $i = a$ and $(\lambda_1, \ldots, \lambda_{a-1}) = (0, 1, \ldots, a-2)$. If $a = 2$, then $\binom{\lambda}{\mu} = \binom{0\ n}{1} \in \mathcal{L}_n$, so we may also assume that $a \geq 3$. If $n - \lambda_a \geq 1$, then

$$\chi(1)/\chi'(1) = \frac{q^{2n}-1}{q^{2\lambda_a}-1} \cdot \prod_{i'=0}^{a-2} \frac{q^{\lambda_a}-q^{i'}}{q^{\lambda_a-1}-q^{i'}} \cdot \prod_{j=1}^{b} \frac{q^{\lambda_a}+q^j}{q^{\lambda_a-1}+q^j} > q^5,$$

and we are again done. Assume $n \leq \lambda_a$. In this case $n = \lambda_a \geq a$. If $a = 3, 4$ then $\binom{\lambda}{\mu} = \binom{0\ 1\ n}{1\ 2}$, $\binom{0\ 1\ 2\ n}{1}$, or $\binom{0\ 1\ 2\ n}{1\ 2\ 3}$, whence $\chi(1) > D(n)$. If $a \geq 5$, then

$$\chi(1)/\chi'(1) \geq \frac{q^n-1}{q^{n-1}-1} \cdot \frac{q^n-q}{q^{n-1}-q} \cdot \frac{q^n-q^2}{q^{n-1}-q^2} \cdot \frac{q^n-q^3}{q^{n-1}-q^3} \cdot \frac{q^n+q}{q^{n-1}+q} > q^5,$$

and we are again done.

6) Finally, we consider the case where $\binom{\lambda}{\mu} = \binom{0,\ 1,\ 2,\ \dots\ ,a-1}{1,\ 2,\ \dots\ ,b}$. Consider the character χ' labelled by the symbol $\binom{\lambda'}{\mu'} = \binom{0,\ 1,\ 2,\ \dots\ ,a-2}{1,\ 2,\ \dots\ ,b-1}$ of rank $n-1$. Observe that $\binom{\lambda'}{\mu'} \notin \mathcal{L}_{n-1}$, whence $\chi'(1) > D(n-1)$. On the other hand, $\chi(1)/\chi'(1) = q^{a+b-2}(q^{2n}-1)/(q^b-1)(q^{a-1}+1) > (q^{2n}-1)/q$, so $\chi(1) > D(n)$ as desired.

(B) We proceed by induction on n. The case $n = 1$ is trivial. The case $n = 2$ can be checked using [E], and the case $(n,q) = (3,2)$ follows from [Atlas]. Hence for the induction step we may assume that $n \geq 3$ and $(n,q) \neq (3,2)$. Assume that χ is labelled by $((s),\chi_u)$, where $C := C_G(s)$ is not equal to G. Set $E(n) := (q^{2n}-1)(q^{2n-2}-1)/(q^2-1)(q+1)$.

1) One can show that $C = D \times Sp_{2m}(q)$, where $D \simeq \prod_{i=1}^{t} GL_{a_i}^{\alpha_i}(q^{k_i})$, $a_i, k_i \in \mathbb{N}$, $\alpha_i = \pm$, $m < n$, and $m + \sum_{i=1}^{t} k_i a_i = n$. Here we consider the case $m \geq 1$. Clearly, $(Sp_{2n-2m}(q) : D)_{2'}$ is at least the smallest nonunipotent degree of $Sp_{2n-2m}(q)$, which is $(q^{2n-2m}-1)/(q+1)$ by induction hypothesis. Hence

$$\chi(1) \geq (G:C)_{2'} = (G : (Sp_{2n-2m}(q) \times Sp_{2m}(q)))_{2'} \cdot (Sp_{2n-2m}(q) : D)_{2'}$$
$$\geq \frac{q^{2n-2m}-1}{q+1} \cdot \prod_{i=1}^{m} \frac{q^{2(n-i+1)}-1}{q^{2i}-1}.$$

In particular, if $1 \leq m \leq n-2$, then $\chi(1) \geq E(n)$. Assume $m = n-1$. Then $D \simeq \mathbb{Z}_{q-\beta}$ with $\beta = \pm$. Observe that there are exactly $q/2-1$, resp. $q/2$, G-conjugacy classes of such elements s for $\beta = +$, resp. for $\beta = -$. If $\chi_u(1) = 1$, then $\chi(1) = (q^{2n}-1)/(q-\beta)$. Assume $\chi_u(1) > 1$. Then

$$\chi_u(1) \geq (q^{n-1}-1)(q^{n-1}-q)/2(q+1)$$

by [LaS]. It follows that $\chi(1) \geq E(n)$ if $n \geq 3$ and $q \geq 4$, or if $n \geq 4$ and $D = \mathbb{Z}_{q-1}$. Assume that $n \geq 4$ and $(q,D) = (2,\mathbb{Z}_{q+1})$. According to part (A), either $\chi_u(1) = (q^{n-1}-\gamma)(q^{n-1}-\gamma q)/2(q+1)$ for some $\gamma = \pm$ (and there is exactly one such a character for each γ) or $\chi_u(1) > (q^{2n-2}-1)/(q^2-1)$. In the former case (B5) holds, and in the latter case $\chi(1) > E(n)$.

Now we may assume that $m = 0$; in particular, $C = D$.

2) Assume $n = 3$; in particular, $q \geq 4$. If $D \neq GL_3^{\gamma}(q)$, then $\chi(1) \geq E(n)$. So $D = GL_3^{\gamma}(q)$. In this case, if $\chi_u(1) > 1$, then $\chi_u(1) > q+1$ and so $\chi(1) > E(n)$. If $\chi_u(1) = 1$, then (B3) holds.

Assume $n = 4$. If $D \neq GL_4^{\gamma}(q)$, then $\chi(1) \geq E(n)$. So $D = GL_4^{\gamma}(q)$. In this case, if $\chi_u(1) > 1$, then $\chi_u(1) > q+1$ and so $\chi(1) > E(n)$. If $\chi_u(1) = 1$, then (B3) holds.

Assume $n \geq 5$. If $D \neq GU_n(q)$, then one can check that $\chi(1) \geq E(n)$. So $D = GU_n(q)$. In this case, if $\chi_u(1) > 1$ or if $n \geq 6$, then again $\chi(1) > E(n)$. If $\chi_u(1) = 1$ and $n = 5$, then (B4) holds.

(C) We have completed the proof of Theorem 6.1, except that we have not identified the set $\{\chi^{\lambda,\mu} \mid \binom{\lambda}{\mu} \in \mathcal{L}_n\}$ with $\{1_G, \alpha_n, \beta_n, \rho_n^1, \rho_n^2\}$ and the set of $q-1$ non-unipotent characters of degree $(q^{2n}-1)/(q\pm 1)$ with $\{\tau_n^i \mid 1 \leq i \leq q/2-1\} \cup \{\zeta_n^i \mid 1 \leq i \leq q/2\}$. If $(n,q) = (3,2)$ or $(4,2)$, then this identification can be established using [Atlas]. Assume $n \geq 3$ and $(n,q) \neq (3,2)$, $(4,2)$. Then these characters are exactly those of degree $\leq (q^{2n}-1)/(q-1)$, and this identification follows immediately by degree comparison. $\square$

Theorem 6.1 immediately yields the following bound result:

Corollary 6.2. *Let $G = Sp_{2n}(q)$ with $n \geq 3$ and $(n,q) \neq (3,2)$. Let*

$$D(n,q) := \begin{cases} (q-1)(q^2+1)(q^3-1), & n=3, \\ (q^{2n}-1)(q^{n-1}-1)(q^{n-1}-q^2)/2(q^4-1), & n \geq 4. \end{cases}$$

Then for any $\chi \in \mathrm{Irr}(G)$, either $\chi(1) \geq D(n,q)$, or χ is one of the $q+4$ characters 1_G, α_n, β_n, ρ_n^1, ρ_n^2, ζ_n^i, τ_n^i. $\square$

Notice that $D(n,q) = \frac{1}{2}q^{4n-6} \cdot (1 + q^{-4} + O(q^{-5}))$.

Corollary 6.3. *Let $G = Sp_{2n}(q)$ with $n \geq 3$ and $(n,q) \neq (3,2)$. The semisimple characters labelled by $s = \mathrm{diag}(\delta^j, \delta^{-j}, I_{2n-2})$ with $1 \leq j \leq q/2-1$ are exactly the characters τ_n^i. The semisimple characters labelled by $s \in G$ conjugate to $\mathrm{diag}(\xi^j, \xi^{-j}, I_{2n-2})$ with $1 \leq j \leq q/2$ are exactly the characters ζ_n^i.*

Proof. Degree comparison using Theorem 6.1(B). $\square$

7. Dual pairs and branching rules for Weil representations

7.1. **The dual pair $Sp_{2n}(q) \times O_2^-(q)$.** We recall the construction [T] of the dual pair $Sp_{2n}(q) \times O_2^-(q)$ in characteristic 2. Let $U = \mathbb{F}_{q^2}^n = \langle a_1, \dots, a_n \rangle_{\mathbb{F}_{q^2}}$ be endowed with standard Hermitian form $\circ$ (that is, semilinear on the first component and that has I_n as a Gram matrix in the given basis). Let $\bar{\alpha} = \alpha^q$ for any $\alpha \in \mathbb{F}_{q^2}$, and $\bar{u} = \sum_{i=1}^n \overline{\alpha}_i a_i$ for any $u = \sum_{i=1}^n \alpha_i a_i \in U$. We can define two nondegenerate symplectic forms, one on the $\mathbb{F}_q$-space U: $(u,v) = u \circ v + v \circ u$, and another on the $\mathbb{F}_2$-space U: $\langle u, v \rangle = \mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_2}((u,v))$. Next we consider

$$E = \mathbb{C}^{q^{2^n}} = \langle e_u \mid u \in U \rangle_{\mathbb{C}}.$$

The dual pair $Sp_{2n}(q) \times O_2^-(q)$ will be defined inside $GL(E)$. For each $v \in U$ consider the following elements of $GL(E)$:

$$f_v \;:\; e_u \mapsto (-1)^{\langle u,v \rangle} e_u, \qquad t_v \;:\; e_u \mapsto e_{u+v}.$$

Then $P := \langle f_v, t_v \mid v \in U \rangle \simeq 2^{1+4nf}$ is an extraspecial 2-subgroup of $GL(E)$. Let

$$\mathcal{N} := \{X \in GL(E) \mid XPX^{-1} = P,\ \det(X) = \pm 1\}.$$

It is known that $\mathcal{N}/Z(\mathcal{N})P \geq \Omega_{4nf}^+(2)$ contains canonical subgroups $\bar{H} \simeq GU_{2n}(q)$ and $\bar{S} \simeq Sp_{2n}(q)$. In particular, if $g \in Sp_{2n}(q) = Sp(U, (\cdot,\cdot))$, then $gf_ug^{-1} = f_{g(u)}$, $gt_ug^{-1} = t_{g(u)}$. Let $\widehat{H}$, resp. $\widehat{S}$, be the complete inverse image of $\bar{H}$, resp. of $\bar{S}$, in $\mathcal{N}$. Fix $\theta \in \mathbb{F}_{q^2}^{\bullet}$ of order $q+1$, and consider the following two elements of $\mathcal{N}$:

$$\boldsymbol{\vartheta} \;:\; e_u \mapsto e_{\theta u}, \qquad \boldsymbol{j} \;:\; e_u \mapsto e_{\bar{u}}.$$

Clearly, $D := \langle \boldsymbol{\vartheta}, \boldsymbol{j} \rangle \simeq O_2^-(q)$. It is shown in [T] that $C_{\widehat{S}}(\boldsymbol{\vartheta}, \boldsymbol{j}) = Z(\mathcal{N}) \times S$, where

$$S = C_{\widehat{S}}(\boldsymbol{\vartheta}, \boldsymbol{j})^{(\infty)} \simeq Sp_{2n}(q), \tag{12}$$

provided that $(n,q) \neq (1,2)$, $(2,2)$, $(3,2)$, $(1,4)$. The subgroup $S \times D$ is the desired dual pair $Sp_{2n}(q) \times O_2^-(q)$. On the other hand, under the same assumption on (n,q), $C_{\widehat{H}}(\boldsymbol{\vartheta}) = Z(\mathcal{N}) \times H$, where $H = O^2(C_{\widehat{H}}(\boldsymbol{\vartheta})) \simeq GU_{2n}(q)$. The subgroup

$H \times \langle \boldsymbol{\vartheta} \rangle$ forms the dual pair $GU_{2n}(q) \times GU_1(q)$; these two dual pairs $H \times \langle \boldsymbol{\vartheta} \rangle$ and $S \times D$ form what is called a pair of *see-saw dual pairs* in the terminology of Kudla [Ku].

Let ω_n denote the character of $S \times D$ acting on E. It is also shown in [T] that $\omega_n|_S = \zeta_n$, and moreover one can label the irreducible characters of D as $\beta = 1_D$, α of degree 1, and ν_i, $1 \leq i \leq q/2$, of degree 2 such that

$$\omega_n|_{S \times D} = \alpha_n \otimes \alpha + \beta_n \otimes \beta + \sum_{i=1}^{q/2} \zeta_n^i \otimes \nu_i. \tag{13}$$

Assume $q \geq 8$, and let $k, l \in \mathbb{N}$ be such that $k + l = n$. We can repeat the above construction but with n replaced throughout by k, resp. by l, and subscript 1, resp. 2, attached to all letters U, E, P, $\mathcal{N}$, S, D, $\boldsymbol{\vartheta}$, and $\boldsymbol{j}$. Thus we get the dual pair $S_1 \times D_1 \simeq Sp_{2k}(q) \times O_2^-(q)$ inside $GL(E_1)$ with character ω_k, and the dual pair $S_2 \times D_2 \simeq Sp_{2l}(q) \times O_2^-(q)$ inside $GL(E_2)$ with character ω_l. Now we can identify U with $U_1 \oplus U_2$. This in turn identifies E with $E_1 \otimes E_2$, P with $P_1 \otimes P_2$, $\boldsymbol{\vartheta}$ with $\boldsymbol{\vartheta}_1 \otimes \boldsymbol{\vartheta}_2$, $\boldsymbol{j}$ with $\boldsymbol{j}_1 \otimes \boldsymbol{j}_2$. This identification also embeds $\mathcal{N}_1 \otimes \mathcal{N}_2$ in $\mathcal{N}$, $\bar{S}_1 \times \bar{S}_2$ in $\bar{S}$, and $\widehat{S}_1 \times \widehat{S}_2$ in $\widehat{S}$. Suppose $g_1 \in S_1$ and $g_2 \in S_2$. Then $g_1 \otimes g_2$ centralizes both $\boldsymbol{\vartheta}$ and $\boldsymbol{j}$, and belongs to $\widehat{S}$. It follows by (12) that

$$S_1 \times S_2 = (S_1 \times S_2)^{(\infty)} < C_{\widehat{S}}(\boldsymbol{\vartheta}, \boldsymbol{j})^{(\infty)} = S.$$

Moreover, if $g := g_1 \otimes g_2$, $x := \boldsymbol{\vartheta}^a \boldsymbol{j}^b$ (for some $a, b \in \mathbb{Z}$), then

$$xg = (\boldsymbol{\vartheta}_1 \otimes \boldsymbol{\vartheta}_2)^a (\boldsymbol{j}_1 \otimes \boldsymbol{j}_2)^b (g_1 \otimes g_2) = \boldsymbol{\vartheta}_1^a \boldsymbol{j}_1^b g_1 \otimes \boldsymbol{\vartheta}_2^a \boldsymbol{j}_2^b g_2 = x_1 g_1 \otimes x_2 g_2,$$

where $x_1 = \boldsymbol{\vartheta}_1^a \boldsymbol{j}_1^b$, $x_2 = \boldsymbol{\vartheta}_2^a \boldsymbol{j}_2^b$. Taking trace and identifying x_1, x_2 with x (which causes no loss in computing ω_n, ω_k, and ω_l) we obtain

Lemma 7.1. *Let $q \geq 8$ and let $Sp_{2k}(q) \times Sp_{2l}(q)$ be a standard subgroup of $Sp_{2n}(q)$ with $k + l = n$. Then $\omega_n(xg) = \omega_k(xg_1) \cdot \omega_l(xg_2)$ for any $x \in O_2^-(q)$ and any $g = (g_1, g_2) \in Sp_{2k}(q) \times Sp_{2l}(q)$.* □

Proposition 7.2. *Let $S = Sp_{2n}(q)$ with $n \geq 3$ and let $H = Sp_{2n-2}(q)$ be a standard subgroup of S. Then*

$$\alpha_n|_H = q\alpha_{n-1} + (q-1)\sum_{k=1}^{q/2} \zeta_{n-1}^k, \;\; \beta_n|_H = q\beta_{n-1} + (q-1)\sum_{k=1}^{q/2} \zeta_{n-1}^k,$$

$$\zeta_n^i|_H = \zeta_{n-1}^i + (q-1)\left(2\sum_{k=1}^{q/2} \zeta_{n-1}^k + \alpha_{n-1} + \beta_{n-1}\right),$$

for $1 \leq i \leq q/2$.

Proof. 1) The standard embedding of H in S can be written as $h \in H \mapsto g = \mathrm{diag}(1, 1, h) \in S$. In the notation of §3 it follows that $\dim_{\mathbb{F}_{q^2}} \mathrm{Ker}(g - \xi^j)$ equals $2 + \dim_{\mathbb{F}_{q^2}} \mathrm{Ker}(h - \xi^j)$ if $j = 0$ and $\dim_{\mathbb{F}_{q^2}} \mathrm{Ker}(h - \xi^j)$ if $j \neq 0$. Hence (4) implies that

$$\zeta_n^i|_H = \sum_{j \neq i,\ k \neq j} \zeta_{n-1}^k = q\zeta_{n-1}^i + (q-1)\sum_{k \neq i} \zeta_{n-1}^k = \zeta_{n-1}^i + (q-1)\sum_{k=0}^{q} \zeta_{n-1}^k$$

for any $i = 0, 1, \dots, q$. In particular, we get the desired formula for $i \neq 0$, since $\zeta_{n-1}^0 = \alpha_{n-1} + \beta_{n-1}$. In order to prove the formulae for $\alpha_n|_H$ and $\beta_n|_H$, it suffices to prove one of them, since we have just shown that

$$\zeta_n^0|_H = q(\alpha_{n-1} + \beta_{n-1}) + 2(q-1)\sum_{k=1}^{q/2} \zeta_{n-1}^k. \tag{14}$$

2) Here we assume that $q \geq 8$. According to (13), β_n is just the S-character of the 1_D-homogeneous component of ω_n, whence

$$\beta_n(g) = \frac{1}{2(q+1)} \sum_{x \in D} \omega_n(xg).$$

If $g = \mathrm{diag}(1,1,h) = 1 \otimes h$ as in 1), then $\omega_n(xg) = \omega_1(x) \cdot \omega_{n-1}(xh)$ by Lemma 7.1. Moreover, if $0 \leq a \leq q$ and $b = 0, 1$, then $\omega_1(\boldsymbol{\vartheta}^a \boldsymbol{j}^b)$ equals q^2, resp. q, or 1, provided that $(a,b) = (0,0)$, resp. $b = 1$, or $(a,b) = (\neq 0, 0)$. It follows that

$$\beta_n(g) = \left(q^2 \omega_{n-1}(h) + \sum_{a=1}^{q} \omega_{n-1}(\boldsymbol{\vartheta}^a h) + q \sum_{a=0}^{q} \omega_{n-1}(\boldsymbol{\vartheta}^a \boldsymbol{j} h) \right) / 2(q+1),$$

$$\beta_{n-1}(h) = \left(\omega_{n-1}(h) + \sum_{a=1}^{q} \omega_{n-1}(\boldsymbol{\vartheta}^a h) + \sum_{a=0}^{q} \omega_{n-1}(\boldsymbol{\vartheta}^a \boldsymbol{j} h) \right) / 2(q+1).$$

Therefore

$$\beta_n(g) - q\beta_{n-1}(h) = \frac{q-1}{2(q+1)} \left(q\omega_{n-1}(h) - \sum_{a=1}^{q} \omega_{n-1}(\boldsymbol{\vartheta}^a h) \right)$$

$$= \frac{q-1}{2(q+1)} \left((q+1)\omega_{n-1}(h) - \sum_{a=0}^{q} \omega_{n-1}(\boldsymbol{\vartheta}^a h) \right) = \frac{q-1}{2} \cdot \omega_{n-1}(h) - \frac{q-1}{2} \cdot \zeta_{n-1}^0(h)$$

$$= \frac{q-1}{2} \left(\alpha_{n-1} + \beta_{n-1} + 2\sum_{k=1}^{q/2} \zeta_{n-1}^k \right)(h) - \frac{q-1}{2}(\alpha_{n-1} + \beta_{n-1})(h)$$

$$= (q-1)\sum_{k=1}^{q/2} \zeta_{n-1}^k(h),$$

and so we are done.

3) Now we consider the case $q = 2, 4$; in particular, $q+1$ is prime. By (14), there are nonnegative integers a, b, c_i with $a, b \leq q$ and $\sum_{i=1}^{q/2} c_i \leq q(q-1)$ such that $\beta_n|_H = a\alpha_{n-1} + b\beta_{n-1} + \sum_{i=1}^{q/2} c_i \zeta_{n-1}^i$. Since the degrees of β_n, β_{n-1}, and ζ_{n-1}^i are divisible by $q^{n-1}+1$, so is $a\alpha_{n-1}(1)$. But $0 \leq a \leq q$, hence $a = 0$. Comparing the degree we see $q(q-b)(q^{n-2}+1) = (2\sum_i c_i - q(q-1))(q^{n-1}-1)$.

Claim that $b = q$ and $\sum_i c_i = q(q-1)/2$. Indeed, the above equality implies that $(q-b)(q+1)$ is divisible by $q^{n-1}-1$. If $n \geq 4$, then it follows that $b = q$. If $n = 3$ but $b \neq q$, then $b = 1$ and $\sum_i c_i = q^2/2$. However this would imply $\beta_3(t) = -(q^4 - q^3 + 2q^2 - q)/2$ for any transvection $t \in H$, contrary to [Lu].

We have shown that $\beta_n|_H = q\beta_{n-1} + \sum_{i=1}^{q/2} c_i \zeta_{n-1}^i$ with $\sum_i c_i = q(q-1)/2$. Recall that we are assuming $q+1$ is prime, so $\Gamma := Gal(\mathbb{Q}(\xi)/\mathbb{Q})$ is cyclic. Formula (4) shows that Γ acts transitively on the set $\{\zeta_{n-1}^i \mid 1 \leq i \leq q/2\}$; indeed, the Galois automorphism $\xi \mapsto \xi^k$ sends ζ_{n-1}^i to ζ_{n-1}^{ik}. On the other hand, β_n, as the only irreducible constituent of degree $(q^n+1)(q^n+q)/2(q+1)$ of the rational-valued character ζ_n, is Γ-stable. Similarly, β_{n-1} is Γ-stable. It therefore follows that $c_1 = \ldots = c_{q/2} = q-1$, as stated. $\square$

In what follows we need the character values at some particular elements of G; namely a transvection (equivalently, a long-root element) $\boldsymbol{t} := \mathrm{diag}\left(\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, I_{2n-2}\right)$, a short-root element $\boldsymbol{s} := \mathrm{diag}\left(\begin{pmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}, I_{2n-4}\right)$, a double-transvection $\boldsymbol{d} := \mathrm{diag}\left(\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, I_{2n-4}\right)$, and semisimple elements

$$\boldsymbol{c}_j := \mathrm{diag}\left(\delta^j, \delta^{-j}, 1, \ldots, 1\right), \qquad 1 \le j \le (q-2)/2,$$

where all matrices are written with respect to the basis

$$(e_1, f_1, e_2, f_2, e_3, f_3, \ldots, e_n, f_n).$$

If we embed the corresponding elements in a standard subgroup $Sp_4(q)$, then $\boldsymbol{t}$, $\boldsymbol{s}$, and $\boldsymbol{d}$ belong to classes A_2, A_{31}, and A_{32} (in the notation of [E]), respectively.

Corollary 7.3. *Let $G = Sp_{2n}(q)$ with $n \ge 2$ and let $1 \le i \le q/2$, $1 \le j \le (q-2)/2$. Then*

$$\alpha_n(\boldsymbol{t}) = \beta_n(\boldsymbol{t}) = -\frac{q^{2n-1}-q}{2(q+1)},\ \zeta_n^i(\boldsymbol{t}) = -\frac{q^{2n-1}+1}{q+1},$$

$$\alpha_n(\boldsymbol{s}) = \frac{q^{2n-2}-q^{n+1}-q^n+q}{2(q+1)},\ \beta_n(\boldsymbol{s}) = \frac{q^{2n-2}+q^{n+1}+q^n+q}{2(q+1)},$$

$$\zeta_n^i(\boldsymbol{s}) = \frac{q^{2n-2}-1}{q+1},$$

$$\alpha_n(\boldsymbol{d}) = \beta_n(\boldsymbol{d}) = \frac{q^{2n-2}+q}{2(q+1)},\ \zeta_n^i(\boldsymbol{d}) = \frac{q^{2n-2}-1}{q+1},$$

$$\alpha_n(\boldsymbol{c}_j) = \frac{q^{2n-2}-q^n-q^{n-1}+q}{2(q+1)},\ \beta_n(\boldsymbol{c}_j) = \frac{q^{2n-2}+q^n+q^{n-1}+q}{2(q+1)},$$

$$\zeta_n^i(\boldsymbol{c}_j) = \frac{q^{2n-2}-1}{q+1}.$$

Proof. The character values of ζ_n^i for $0 \le i \le q/2$ are computed using (4). In particular, we know the character values of $\zeta_n^0 = \alpha_n + \beta_n$. We will deal with α_n and β_n by induction on n. The induction base $n = 2$ can be checked using [E]. For the induction step assume $n \ge 3$. We may also assume that $\boldsymbol{t}, \boldsymbol{s}, \boldsymbol{d}, \boldsymbol{c}_j$ are contained in a standard subgroup $H \simeq Sp_{2n-2}(q)$. By Proposition 7.2, $(\beta_n - \alpha_n)(x) = q(\beta_{n-1} - \alpha_{n-1})(x)$ for any $x \in \{\boldsymbol{t}, \boldsymbol{s}, \boldsymbol{d}, \boldsymbol{c}_j\}$. Since we know $\alpha_n + \beta_n$ and we know $(\beta_{n-1} - \alpha_{n-1})(x)$ by induction, the stated formulae follow easily. $\square$

Recall that for any Q_1-character $\lambda \in \mathcal{O}_2^{\varepsilon}$, $K_\lambda := Stab_{P_1}(\lambda) = Q_1 : I_\lambda$, where $I_\lambda := Stab_{L_1}(\lambda)$ equals $J_\lambda := Stab_{L_1'}(\lambda) \simeq O_{2n-2}^{\varepsilon}(q)$. Let $\hat{\kappa}$ be the character of K_λ, where $\hat{\kappa}|_{Q_1} = \lambda$ and $\hat{\kappa}|_{I_\lambda} = \kappa$. In what follows, the P_1-module $\boldsymbol{B}_\varepsilon$ is defined to be $\mathrm{Ind}_{K_\lambda}^{P_1}(\hat{\kappa})$. In the next statement, we will consider $\alpha_{n-1} \otimes 1_{T_1}$ as a character of $L_1 = L_1' \times T_1$, and then inflate it to $P_1 = Q_1 L_1$. Similarly for $\beta_{n-1} \otimes 1_{T_1}$, $\zeta_{n-1}^i \otimes 1_{T_1}$.

Proposition 7.4. *Let $G = Sp_{2n}(q)$ with $n \geq 3$, $(n,q) \neq (3,2)$, $(4,2)$, and let $1 \leq i \leq q/2$. Then*

$\alpha_n|_{P_1} = \alpha_{n-1} \otimes 1_{T_1} + \boldsymbol{B}_-$, $\beta_n|_{P_1} = \beta_{n-1} \otimes 1_{T_1} + \boldsymbol{B}_+$, $\zeta^i_n|_{P_1} = \zeta^i_{n-1} \otimes 1_{T_1} + \boldsymbol{B}_+ + \boldsymbol{B}_-$.

Proof. 1) Let $\phi \in \{\alpha_n, \beta_n, \zeta^i_n\}$ and let V be a module affording ϕ. By Corollary 4.2, ϕ satisfies $(\mathcal{W}_2^-)$, whence ϕ satisfies $(\mathcal{W}_1)$ by Proposition 5.7. It follows that ϕ satisfies the conclusion of Theorem 5.11(i). Thus there are nonnegative integers c, d such that $V|_{P_1} = C_V(Q_1) + c\boldsymbol{B}_+ + d\boldsymbol{B}_-$ and moreover $(c,d) \neq (0,0)$. By Proposition 7.2, there are nonnegative integers a, b, e such that

$$\dim(C_V(Q_1)) = a\alpha_{n-1}(1) + b\beta_{n-1}(1) + e\zeta^1_{n-1}(1). \tag{15}$$

First we consider the case $\phi = \alpha_n$. Observe that $\alpha_n(1) = \alpha_{n-1}(1) + \boldsymbol{B}_-(1) < 2\boldsymbol{B}_-(1) < \boldsymbol{B}_-(1) + \boldsymbol{B}_+(1)$, and $\boldsymbol{B}_+(1) < \alpha_n(1) < \alpha_{n-1}(1) + \boldsymbol{B}_+(1)$ if $n \geq 4$ and $(n,q) \neq (4,2)$, and $\alpha_n(1) < \boldsymbol{B}_+(1)$ if $n = 3$. It follows that $(c,d) = (0,1)$ and $C_V(Q_1)$ affords the L'_1-character α_{n-1}. Thus we have proved the desired formula for $\alpha_n|_{P'_1}$. In particular,

$$\boldsymbol{B}_-|_{L'_1} = (q-1)(\alpha_{n-1} + \sum_{k=1}^{q/2} \zeta^k_{n-1}). \tag{16}$$

Next we consider the case $\phi = \beta_n$. Since $\beta_n(1) < 2\boldsymbol{B}_-(1) < \boldsymbol{B}_+(1) + \boldsymbol{B}_-(1)$, $\{c,d\} = \{0,1\}$. Assume $d = 1$. Then $\beta_n|_{L'_1}$ contains α_{n-1} by (16), contrary to Proposition 7.2. Hence $(c,d) = (1,0)$. Now $\dim(C_V(Q_1)) = \beta_{n-1}(1)$, whence (15) implies that $(a,b,e) = (0,1,0)$ (since $n \geq 3$ and $(n,q) \neq (3,2)$, $(4,2)$). Thus we have proved the desired formula for $\beta_n|_{P'_1}$. In particular,

$$\boldsymbol{B}_+|_{L'_1} = (q-1)(\beta_{n-1} + \sum_{k=1}^{q/2} \zeta^k_{n-1}). \tag{17}$$

Now we consider the case $\phi = \zeta^i_n$. Applying Proposition 7.2 and (16), (17) to $\zeta^i_n|_{P_1} = C_V(Q_1) + c\boldsymbol{B}_+ + d\boldsymbol{B}_-$, we get

$$\zeta^i_{n-1} + (q-1)(\alpha_{n-1} + \beta_{n-1} + 2\sum_{k=1}^{q/2} \zeta^k_{n-1})$$
$$= C_V(Q_1) + c(q-1)(\beta_{n-1} + \sum_{k=1}^{q/2} \zeta^k_{n-1}) + d(q-1)(\alpha_{n-1} + \sum_{k=1}^{q/2} \zeta^k_{n-1}).$$

Since all α_{n-1}, β_{n-1}, and ζ^k_{n-1} are irreducible and distinct, we see that $c \leq 1$ and $d \leq 1$. If $(c,d) = (1,0)$, then α_{n-1} appears in $C_V(Q_1)$ but not in $[V, Q_1]$, contrary to Lemma 2.1. Similarly, $(c,d) \neq (0,1)$. Thus $c = d = 1$, and $C_V(Q_1) = \zeta^i_{n-1}$ as L'_1-module, and so we have proved the desired formula for $\zeta^i_n|_{P'_1}$.

2) Consider any $g \in P_1 \setminus P'_1$. Write $g = \begin{pmatrix} \delta^j & * & * \\ 0 & h & * \\ 0 & 0 & \delta^{-j} \end{pmatrix}$ in the basis $(e_1, \ldots, e_n, f_2, \ldots, f_n, f_1)$, for some $\delta^j \neq 1$ and $h \in L'_1$. For any $\lambda \in \mathcal{O}^\varepsilon_2$, $Stab_{P_1}(\lambda) < P'_1$, whence $\boldsymbol{B}_+(g) = \boldsymbol{B}_-(g) = 0$. Therefore, in order to finish the proof of the proposition, it suffices to show that $\zeta^i_n(g) = \zeta^i_{n-1}(h)$ for any $i = 0, 1, \ldots, q/2$. Indeed, for any ξ^k we have $\delta^{\pm j} \neq \xi^k$, and so direct computation shows that

$$|\{v \in W \otimes_{\mathbb{F}_q} \mathbb{F}_{q^2} \mid g(v) = \xi^k v\}| = |\{w \in \langle e_2, \ldots, e_n, f_2, \ldots, f_n \rangle_{\mathbb{F}_{q^2}} \mid h(w) = \xi^k w\}|,$$

i.e. $\dim_{\mathbb{F}_{q^2}} \mathrm{Ker}(g-\xi^k) = \dim_{\mathbb{F}_{q^2}} \mathrm{Ker}(h-\xi^k)$. Now (4) yields that $\zeta_n^i(g) = \zeta_{n-1}^i(h)$, as desired. □

Corollary 7.5. *Let $G = Sp_{2n}(q)$ with $n \geq 2$. Write $q+1 = \ell^a r$ with $(r,\ell) = 1$.*

(i) *Let φ be any nontrivial composition factor of $\widehat{\alpha}_n$, $\widehat{\beta}_n$, or $\hat{\zeta}_n^i$. Then φ is of form $\rho - a \cdot 1_G$, where $\rho \in \{\widehat{\alpha}_n, \widehat{\beta}_n, \hat{\zeta}_n^j \mid 1 \leq j \leq (r-1)/2\}$, and $a = 0$ or 1. Moreover, $a = 1$ if and only if $\ell|(q+1)$ and $\rho = \widehat{\beta}_n$.*

(ii) *The Brauer characters $\widehat{\alpha}_n$, $\widehat{\beta}_n$, $\hat{\zeta}_n^j$ with $1 \leq j \leq (r-1)/2$ each have a unique nontrivial irreducible constituent, and these $(r+3)/2$ constituents are paiwise distinct.*

Proof. (i) The statement for $n = 2$, resp. for $(n,q) = (3,2)$ and $(4,2)$, can be checked directly using [Wh1], resp. [JLPW]. We will proceed by induction on n and assume $n \geq 3$ and $(n,q) \neq (3,2), (4,2)$.

1) It was established in the proof of [DT, Theorem 7.2] that

$$\hat{\zeta}_n^i - \hat{\zeta}_n^j = \delta_{i,0} - \delta_{j,0}, \tag{18}$$

whenever $0 \leq i,j \leq q$ and $i \equiv j (\mathrm{mod}\, r)$. Together with (5) and (6), (18) implies that any nontrivial composition factor of $\hat{\zeta}_n^i$ is a composition factor of $\widehat{\alpha}_n$, $\widehat{\beta}_n$, or $\hat{\zeta}_n^j$ with $1 \leq j \leq (r-1)/2$.

2) Consider the semisimple character ϕ_j labelled by $s_j \in G$ conjugate to $\mathrm{diag}(\xi^j, \xi^{-j}, I_{2n-2})$ with $1 \leq j \leq (r-1)/2$. Since the ℓ'-part s'_j of s_j is nontrivial semisimple and $\phi_j(1) = (q^{2n}-1)/(q+1) = (G : C_G(s'_j))_{2'}$, [HM, Proposition 1] implies that $\widehat{\phi}_j$ is irreducible. Moreover, these $(r-1)/2$ characters ϕ_j belong to $(r-1)/2$ distinct nonunipotent ℓ-blocks, since the s'_j are pairwise nonconjugate and nontrivial. On the other hand, by Corollary 6.3, $\{\phi_j \mid 1 \leq j \leq (r-1)/2\} \subseteq \{\zeta_n^i \mid 1 \leq i \leq q/2\}$. The result of 1) shows that there are at most $(r-1)/2$ distinct irreducible Brauer characters of degree $(q^{2n}-1)/(q+1)$ that can arise as composition factors of $\hat{\zeta}_n^i$. We conclude that the Brauer characters $\hat{\zeta}_n^j$ with $1 \leq j \leq (r-1)/2$ are irreducible and pairwise distinct.

3) Since α_n meets the Landazuri-Seitz-Zalesskii bound $d(G)$, $\widehat{\alpha}_n$ is irreducible. Next assume φ is a nontrivial composition factor of $\widehat{\beta}_n$ and $\varphi \neq \widehat{\beta}_n$. Clearly, φ cannot be trivial on Q_1. On the other hand, $\beta_n|_{P_1} = \beta_{n-1} \otimes 1_{T_1} + \boldsymbol{B}_+$ by Proposition 7.4, and $\widehat{\boldsymbol{B}}_+$ is irreducible over P_1. It follows that $\varphi|_{P_1}$ contains $\widehat{\boldsymbol{B}}_+$. Now any composition factor of $\widehat{\beta}_n - \varphi$ is trivial on Q_1 and therefore equal to 1_G. Thus $\varphi = \widehat{\beta}_n - a \cdot 1_G$ with $a \geq 1$. Let ψ be any P_1-composition factor of $\varphi - \widehat{\boldsymbol{B}}_+$. Then ψ is trivial on Q_1 and $\psi|_{L'_1}$ is contained in $\widehat{\beta}_{n-1}$. By induction hypothesis and by the assumption on φ, $\psi|_{L'_1}$ equals $1_{L'_1}$ or $\widehat{\beta}_{n-1} - 1_{L'_1}$. If the first possibility realizes for all such ψ, then we come to the conclusion that the Brauer character $\widehat{\beta}_n|_{P_1} - \widehat{\boldsymbol{B}}_+$ is trivial on P_1, contrary to Proposition 7.4. Hence the second possibility must occur for at least one such ψ, whence $\ell|(q+1)$ and $\varphi = \widehat{\beta}_n - 1_G$.

Conversely, assume $\ell|(q+1)$. Then (6) and (18) imply that $\widehat{\alpha}_n + \widehat{\beta}_n = \hat{\zeta}_n^0|_G = 1_G + \hat{\zeta}_n^r$, whence 1_G must occur either in $\widehat{\alpha}_n$ or $\widehat{\beta}_n$. But $\widehat{\alpha}_n$ is irreducible, hence 1_G enters $\widehat{\beta}_n$ and the above discussion shows that $\widehat{\beta}_n - 1_G$ is irreducible.

(ii) follows from (i) and degree comparison. □

7.2. **The dual pair $Sp_{2n}(q) \times O_2^+(q)$.** We recall the construction [T] of the dual pair $Sp_{2n}(q) \times O_2^+(q)$ in characteristic 2. Let $U = \mathbb{F}_q^{2n}$ be endowed with standard symplectic form $(\cdot,\cdot)$. We will also consider the $\mathbb{F}_2$-symplectic form $\langle u,v\rangle = \mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_2}((u,v))$ on U, and let

$$E = \mathbb{C}^{q^{2^n}} = \langle e_u \mid u \in U\rangle_{\mathbb{C}}.$$

Clearly, $S := Sp_{2n}(q)$ acts on E via $g\ :\ e_u \mapsto e_{g(u)}$. Fix $\delta \in \mathbb{F}_q^\bullet$ of order $q-1$, and consider the following endomorphisms of E:

$$\boldsymbol{\delta}\ :\ e_u \mapsto e_{\delta u}$$

(for any $u \in U$) and

$$\boldsymbol{j}\ :\ e_0 \mapsto e_0,\ e_v \mapsto \frac{1}{q^n+1} \sum_{0\neq w\in U,\ \langle v,w\rangle=0} e_w - \frac{q^n+2}{q^n(q^n+1)} \sum_{w\in U,\ \langle w,v\rangle\neq 0} e_w$$

(for any $0 \neq v \in U$). One can check that $D := \langle \boldsymbol{\delta}, \boldsymbol{j}\rangle \simeq O_2^+(q)$ and that D centralizes S. The subgroup $S \times D$ of $GL(E)$ is the desired dual pair $Sp_{2n}(q) \times O_2^+(q)$. Let ω_n denote the character of $S \times D$ acting on E. It is shown in [T] that $\omega_n|_S = \tau_n$, and moreover one can label the irreducible characters of D as $\beta = 1_D$, α of degree 1, and μ_i, $1 \le i \le (q-2)/2$, of degree 2 such that

$$\omega_n|_{S\times D} = (\rho_n^1 + 1_S)\otimes\alpha + (\rho_n^2 + 1_S)\otimes\beta + \sum_{i=1}^{(q-2)/2} \tau_n^i \otimes \mu_i. \tag{19}$$

We can repeat the above construction but with n replaced throughout by $n-1$, resp. by 1, and subscript 1, resp. 2, attached to all letters U, E, S, D, $\boldsymbol{\delta}$, and $\boldsymbol{j}$. Thus we get the dual pair $S_1 \times D_1 \simeq Sp_{2n-2}(q) \times O_2^+(q)$ inside $GL(E_1)$ with character ω_{n-1}, and the dual pair $S_2 \times D_2 \simeq Sp_2(q) \times O_2^+(q)$ inside $GL(E_2)$ with character ω_1. Now we can identify U with $U_1 \oplus U_2$. This in turn identifies E with $E_1 \otimes E_2$ and $\boldsymbol{\delta}$ with $\boldsymbol{\delta}_1 \otimes \boldsymbol{\delta}_2$. This identification also embeds $S_1 \otimes S_2$ in S. In what follows, we denote $x_1 := \boldsymbol{\delta}_1^a \boldsymbol{j}_1^b$ and $x_2 := \boldsymbol{\delta}_2^a \boldsymbol{j}_2^b$ for $x = \boldsymbol{\delta}^a \boldsymbol{j}^b$.

Lemma 7.6. *Let $Sp_{2n-2}(q) \times Sp_2(q)$ be a standard subgroup of $Sp_{2n}(q)$. Then $\omega_n(gx) = \omega_{n-1}(hx_1)\cdot\omega_1(x_2)$ for any $x \in O_2^+(q)$ and any $g = h \otimes 1 \in Sp_{2n-2}(q) \times Sp_2(q)$.*

Proof. Suppose first that $x = \boldsymbol{\delta}^a$. Then $gx = (h\otimes 1)(\boldsymbol{\delta}_1\otimes\boldsymbol{\delta}_2)^a = h\boldsymbol{\delta}_1^a \otimes \boldsymbol{\delta}_2^a = hx_1 \otimes x_2$, whence the statement follows by taking trace.

Next assume that $x = \boldsymbol{\delta}^a\boldsymbol{j}$. Then $\omega_1(x_2) = q$. Let

$$\begin{aligned} N :=&\quad |\{w \in U \mid w \neq 0,\ \langle \boldsymbol{\delta}^{-a} g^{-1}(w), w\rangle = 0\}|,\\ N_1 :=&\quad |\{w \in U_1 \mid w \neq 0,\ \langle \boldsymbol{\delta}_1^{-a} h^{-1}(w), w\rangle = 0\}|. \end{aligned}$$

Then $\omega_n(gx) = 2q^{-n}(N+1) - q^n$, $\omega_{n-1}(hx_1) = 2q^{1-n}(N_1+1) - q^{n-1}$. Direct computation shows that $N + 1 = q^2(N_1 + 1)$, whence $\omega_n(gx) = q\omega_{n-1}(hx_1)$ as stated. $\square$

Proposition 7.7. *Let $S = Sp_{2n}(q)$ with $n \ge 2$, and let $H = Sp_{2n-2}(q)$ be a standard subgroup of S. Then, for $1 \le i \le q/2 - 1$,*

$$\begin{aligned} \rho_n^1|_H =&\quad (q+1)\cdot 1_H + (q+1)\rho_{n-1}^1 + \rho_{n-1}^2 + (q+1)\textstyle\sum_{k=1}^{(q-2)/2} \tau_{n-1}^k,\\ \rho_n^2|_H =&\quad (q+1)\cdot 1_H + (q+1)\rho_{n-1}^2 + \rho_{n-1}^1 + (q+1)\textstyle\sum_{k=1}^{(q-2)/2} \tau_{n-1}^k,\\ \tau_n^i|_H =&\quad \tau_{n-1}^i + (q+1)\left(2\cdot 1_H + 2\textstyle\sum_{k=1}^{(q-2)/2} \tau_{n-1}^k + \rho_{n-1}^1 + \rho_{n-1}^2\right). \end{aligned}$$

Proof. 1) The standard embedding of H in S can be written as $h \in H \mapsto g = h \otimes 1 \in S$. In the notation of §3 it follows that $\dim_{\mathbb{F}_q} \mathrm{Ker}(g - \delta^j)$ equals $2 + \dim_{\mathbb{F}_q} \mathrm{Ker}(h - \delta^j)$ if $j = 0$ and $\dim_{\mathbb{F}_q} \mathrm{Ker}(h - \delta^j)$ if $j \neq 0$. Hence (1) implies that

$$\tau_n^i|_H = \tau_{n-1}^i + (q+1)\tau_{n-1} = \tau_{n-1}^i + (q+1)(2 \cdot 1_H + 2 \sum_{k=1}^{(q-2)/2} \tau_{n-1}^k + \rho_{n-1}^1 + \rho_{n-1}^2)$$

for any $i = 1, \ldots, q-2$. Similarly,

$$\tau_n^0|_H = 2(q+1) \cdot 1_H + (q+2)(\rho_{n-1}^1 + \rho_{n-1}^2) + 2(q+1) \sum_{k=1}^{(q-2)/2} \tau_{n-1}^k. \tag{20}$$

2) According to (19), $\rho_n^2 + 1_S$ is just the S-character of the 1_D-homogeneous component of ω_n, whence

$$\rho_n^2(g) = \frac{1}{2(q-1)} \sum_{x \in D} \omega_n(gx) - 1.$$

If $g = h \otimes 1$ as in 1), then $\omega_n(gx) = \omega_{n-1}(hx) \cdot \omega_2(x)$ by Lemma 7.6. Here we have identified x_1 and x_2 with no loss. Moreover, if $0 \leq a \leq q$ and $b = 0, 1$, then $\omega_2(\boldsymbol{\delta}^a \boldsymbol{j}^b)$ equals q^2, resp. q, or 1, provided that $(a, b) = (0, 0)$, resp. $b = 1$, or $(a, b) = (\neq 0, 0)$. It follows that

$$\rho_n^2(g) = \left(q^2 \omega_{n-1}(h) + \sum_{a=1}^{q-2} \omega_{n-1}(h\boldsymbol{\delta}^a) + q \sum_{a=0}^{q-2} \omega_{n-1}(h\boldsymbol{\delta}^a \boldsymbol{j}) \right) / 2(q-1) - 1,$$

$$\rho_{n-1}^2(h) = \left(\omega_{n-1}(h) + \sum_{a=1}^{q-2} \omega_{n-1}(h\boldsymbol{\delta}^a) + \sum_{a=0}^{q-2} \omega_{n-1}(h\boldsymbol{\delta}^a \boldsymbol{j}) \right) / 2(q-1) - 1.$$

Therefore

$$\begin{aligned} \rho_n^2(g) - q\rho_{n-1}^2(h) &= \frac{1}{2} \left(q\omega_{n-1}(h) - \sum_{a=1}^{q-2} \omega_{n-1}(h\boldsymbol{\delta}^a) \right) + (q-1) \\ = \frac{1}{2} \left((q+1)\omega_{n-1}(h) - (\sum_{a=0}^{q-2} \omega_{n-1}(\boldsymbol{\delta}^a h) - 2) \right) &= \frac{q+1}{2} \cdot \omega_{n-1}(h) - \frac{q-1}{2} \cdot \tau_{n-1}^0(h) \\ = \frac{q+1}{2} \left(2 \cdot 1_H + \rho_{n-1}^1 + \rho_{n-1}^2 + 2 \sum_{k=1}^{(q-2)/2} \tau_{n-1}^k \right)(h) &- \frac{q-1}{2}(\rho_{n-1}^1 + \rho_{n-1}^2)(h) \\ &= (q+1) \cdot 1_H + \rho_{n-1}^1 + \rho_{n-1}^2 + (q+1) \sum_{k=1}^{(q-2)/2} \tau_{n-1}^k(h). \end{aligned}$$

Thus we have proved the branching formula for ρ_n^2, and so we are also done for ρ_n^1 because of (20). $\square$

Corollary 7.8. *Let $G = Sp_{2n}(q)$ with $n \geq 2$ and let $1 \leq i, j \leq q/2 - 1$. Then*

$$\rho_n^1(\boldsymbol{t}) = \rho_n^2(\boldsymbol{t}) = \frac{q^{2n-1}-q}{2(q-1)},\ \tau_n^i(\boldsymbol{t}) = \frac{q^{2n-1}-1}{q-1},$$

$$\rho_n^1(\boldsymbol{s}) = \frac{q^{2n-2}-q^{n+1}+q^n-q}{2(q-1)},\ \rho_n^2(\boldsymbol{s}) = \frac{q^{2n-2}+q^{n+1}-q^n-q}{2(q-1)},\ \tau_n^i(\boldsymbol{s}) = \frac{q^{2n-2}-1}{q-1},$$

$$\rho_n^1(\boldsymbol{d}) = \rho_n^2(\boldsymbol{d}) = \frac{q^{2n-2}-q}{2(q-1)},\ \tau_n^i(\boldsymbol{d}) = \frac{q^{2n-2}-1}{q-1},$$

$$\rho_n^1(\boldsymbol{c}_j) = \frac{(q^{n-1}+1)(q^{n-1}-q)}{2(q-1)} + 1,\ \rho_n^2(\boldsymbol{c}_j) = \frac{(q^{n-1}-1)(q^{n-1}+q)}{2(q-1)} + 1,$$

$$\tau_n^i(\boldsymbol{c}_j) = \frac{q^{2n-2}-1}{q-1} + \tilde{\delta}^{ij} + \tilde{\delta}^{-ij}.$$

Proof. Argue as in the proof of Corollary 7.3, using Proposition 7.7. □

For any linear Q_1-character λ, let $\hat{\lambda}$ be the character of $K_\lambda = Q_1 : I_\lambda$, where $\hat{\lambda}|_{Q_1} = \lambda$ and $\hat{\lambda}|_{I_\lambda} = 1_{I_\lambda}$. In what follows, the P_1-module $\boldsymbol{A}_\varepsilon$ is defined to be $\mathrm{Ind}_{K_\lambda}^{P_1}(\hat{\lambda})$ for $\lambda \in \mathcal{O}_2^\varepsilon$, and the P_1-module $\boldsymbol{C}$ is defined to be $\mathrm{Ind}_{K_\lambda}^{P_1}(\hat{\lambda})$ for $\lambda \in \mathcal{O}_1$. Let δ_j be the linear character of $P_1/P_1' \simeq T_1 = \langle c_1 \rangle$ that sends c_1 to $\tilde{\delta}^j$. In the next statement, we will consider $\rho_{n-1}^1 \otimes 1_{T_1}$ as a character of $L_1 = L_1' \times T_1$, and then inflate it to $P_1 = Q_1 L_1$. Similarly for $\rho_{n-1}^2 \otimes 1_{T_1}$ and $\tau_{n-1}^i \otimes 1_{T_1}$. We will also consider δ_j as a character of P_1.

Proposition 7.9. *Let $G = Sp_{2n}(q)$ with $n \geq 3$, $(n,q) \neq (3,2), (4,2)$, and let $1 \leq i \leq q/2 - 1$. Then*

$$\rho_n^1|_{P_1} = 1_{P_1} + \rho_{n-1}^1 \otimes 1_{T_1} + \boldsymbol{C} + \boldsymbol{A}_-\ , \qquad \rho_n^2|_{P_1} = 1_{P_1} + \rho_{n-1}^2 \otimes 1_{T_1} + \boldsymbol{C} + \boldsymbol{A}_+\ ,$$

$$\tau_n^i|_{P_1} = \delta_i + \delta_{-i} + \tau_{n-1}^i \otimes 1_{T_1} + \boldsymbol{C}(\delta_i + \delta_{-i}) + \boldsymbol{A}_+ + \boldsymbol{A}_-\ .$$

Proof. Define the following Q_1-characters:

$$\omega_1 := \sum_{\lambda \in \mathcal{O}_1} \lambda,\ \omega_2^+ := \sum_{\lambda \in \mathcal{O}_2^+} \lambda,\ \omega_2^- := \sum_{\lambda \in \mathcal{O}_2^-} \lambda.$$

1) Here we show that $\tau_n^0|_{P_1}$ contains $2\boldsymbol{C}$; in particular, $(\rho_n^1 + \rho_n^2)_{Q_1}$ contains $2\omega_1$. Recall that $Q_1 = \{[A, C] \mid A \in \mathbb{F}_q^{2n-2}, C \in \mathbb{F}_q\}$ (with respect to the basis $(e_1, \ldots, e_n, f_2, \ldots, f_n, f_1)$), and we can write any character $\lambda \in \mathcal{O}_1$ in the form $\lambda = \lambda_B\ :\ [A, C] \mapsto (-1)^{\mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_2}({}^t b J_{n-1} A)}$ for some $0 \neq b \in \mathbb{F}_q^{2n-2}$. For such $\lambda = \lambda_B$ we have $K_\lambda := Stab_{P_1}(\lambda) = Q_1 : I_\lambda$, where

$$I_\lambda = \{\mathrm{diag}(x, h, x^{-1}) \mid x \in \mathbb{F}_q^\bullet,\ h \in L_1',\ h(b) = xb\}.$$

Identifying $\mathbb{F}_q^{2n-2}$ with $\langle e_2, \ldots, f_n, f_2, \ldots, f_n \rangle_{\mathbb{F}_q}$ and choosing b to be e_2, we see that

$$K_\lambda = \{g \in G \mid g(e_1) = xe_1,\ g(e_2) \equiv xe_2 (\mathrm{mod}\ \langle e_1 \rangle_{\mathbb{F}_q}),\ x \in \mathbb{F}_q^\bullet\}.$$

By Mackey's formula and Frobenius' reciprocity,

$$(\mathrm{Ind}_{P_1}^G(1_{P_1})|_{P_1}, \boldsymbol{C})_{P_1} = (1_{P_1}, \mathrm{Ind}_{P_1}^G(\boldsymbol{C})|_{P_1})|_{P_1} = \sum_{g \in P_1 \backslash G / K_\lambda} (1_{P_1^g \cap K_\lambda}, \hat{\lambda}|_{P_1^g \cap K_\lambda})_{P_1^g \cap K_\lambda}.$$

Choose $g_1, g_2 \in G$ such that $g_1(e_1) = f_1$ and $g_2(e_1) = e_2$, and consider the following two conjugates $P_{11} := P_1^{g_1}$, $P_{12} := P_1^{g_2}$ of P_1. Observe that the double cosets

$P_1g_1K_\lambda$ and $P_1g_2K_\lambda$ are distinct. One can also check that $K_\lambda \cap P_{11} = I_\lambda \leq \mathrm{Ker}(\hat{\lambda})$ and $K_\lambda \cap P_{12} \leq \mathrm{Ker}(\hat{\lambda})$. It follows that $(\mathrm{Ind}_{P_1}^G(1_{P_1})|_{P_1}, \boldsymbol{C})_{P_1} \geq 2$, as stated.

2) Notice that $\tau_n^i = \mathrm{Ind}_{P_1}^G(\delta_i)$. So by Mackey's formula and Frobenius' reciprocity, for any j we have

$$\begin{aligned}(\tau_n^i|_{P_1}, \boldsymbol{C}\delta_j)_{P_1} &= (\mathrm{Ind}_{P_1}^G(\delta_i)|_{P_1}, \mathrm{Ind}_{K_\lambda}^{P_1}(\hat{\lambda}\delta_i))_{P_1}\\ &= \sum_{g \in P_1 \backslash G / K_\lambda} (\delta_i^g|_{P_1^g \cap K_\lambda}, \hat{\lambda}\delta_j|_{P_1^g \cap K_\lambda})_{P_1^g \cap K_\lambda}.\end{aligned}$$

Consider the elements g_1 and g_2 as in 1). For $j = i$ observe that $\delta_i^{g_2}$ and $\hat{\lambda}\delta_i$ agree on $P_{12} := P_1^{g_2}$. For $j = -i$ observe that $\delta_i^{g_1}$ and $\hat{\lambda}\delta_{-i}$ agree on $P_{11} := P_1^{g_1}$. It follows that $\tau_n^i|_{P_1}$ contains both $\boldsymbol{C}\delta_i$ and $\boldsymbol{C}\delta_{-i}$. Also notice that the last two characters are distinct, as they take different values at $\mathrm{diag}(\delta, \delta, I_{n-2}, \delta^{-1}, I_{n-2}, \delta^{-1})$.

3) Observe that $\rho_n^1(1) < \rho_n^2(1) < 2 \cdot |\mathcal{O}_1| + |\mathcal{O}_2^-|$ and $|\mathcal{O}_2^-| < |\mathcal{O}_2^+|$. On the other hand, $\rho_n^1|_{Q_1}$ contains either ω_2^- or ω_2^+; similarly for $\rho_n^2|_{Q_1}$. It follows that neither $\rho_n^1|_{Q_1}$ nor $\rho_n^2|_{Q_1}$ can contain $2\omega_1$. Together with 1), this implies that the multiplicity of $\boldsymbol{C}$ in $\rho_n^1|_{P_1}$ and $\rho_n^2|_{P_1}$ is exactly 1 (since $\boldsymbol{C}|_{Q_1} = \omega_1$). Also by Frobenius' reciprocity, the multiplicity of 1_{P_1} in $\rho_n^1|_{P_1}$ and $\rho_n^2|_{P_1}$ is exactly 1.

4) Let V be a module affording the character ρ_n^1, and write $V = C_V(Z_1) \oplus [Z_1, V]$. The character of the P_1-module $[Z_1, V]$ can be written as $\phi_+ + \phi_-$, where $\phi_+|_{Q_1} = a\omega_2^+$ and $\phi_-|_{Q_1} = b\omega_2^-$ for $0 \leq a, b \in \mathbb{Z}$. For any hyperplane H of type ε of Q_1 there are $(q-1)$ characters $\lambda \in \mathcal{O}_2^\varepsilon$ that are trivial on H, and their restrictions to $Z_1 \simeq Q_1/H$ exhaust $\mathrm{Irr}(Z_1) \setminus \{1_{Z_1}\}$. It follows that

$$\phi_+(\boldsymbol{t}) = -\frac{\phi_+(1)}{q-1} = -aq^{n-1}(q^{n-1}+1)/2, \ \phi_-(\boldsymbol{t}) = -\frac{\phi_-(1)}{q-1} = -bq^{n-1}(q^{n-1}-1)/2$$

for any transvection $\boldsymbol{t} \in Z_1$. Thus

$$\begin{aligned}\rho_n^1(1) &= \dim(C_V(Z_1)) + aq^{n-1}(q^{n-1}+1)(q-1)/2 + bq^{n-1}(q^{n-1}-1)(q-1)/2,\\ (q^{2n-1}-q)/2(q-1) &= \rho_n^1(\boldsymbol{t})\\ &= \dim(C_V(Z_1)) - aq^{n-1}(q^{n-1}+1)/2 - bq^{n-1}(q^{n-1}-1)/2.\end{aligned}$$

Therefore $a(q^{n-1}+1) + b(q^{n-1}-1) = q^{n-1}-1$, i.e. $(a,b) = (0,1)$. Also, $\dim(C_V(Z_1)) = 1 + \rho_{n-1}^1(1) + |\mathcal{O}_1|$. Since the Q_1-module $C_V(Z_1)$ contains the character ω_1 by 3), Proposition 7.7 now implies that the P_1'-module $C_V(Z_1)$ affords the character $1 + \rho_{n-1}^1 + \boldsymbol{C}|_{P_1'}$. By Corollary 4.2, Proposition 5.7, and Theorem 5.11, for any $\lambda \in \mathcal{O}_2^-$, K_λ acts on the λ-homogeneous component via $\hat{\lambda}$, whence $\phi_- = \boldsymbol{A}_-$. We have shown that

$$\rho_n^1|_{P_1} = 1_{P_1} + \rho_{n-1}^1 \otimes \alpha + \boldsymbol{C} + \boldsymbol{A}_-$$

for some linear character α of T_1.

In the same fashion we can show that

$$\rho_n^2|_{P_1} = 1_{P_1} + \rho_{n-1}^2 \otimes \beta + \boldsymbol{C} + \boldsymbol{A}_+$$

for some linear character β of T_1.

5) From 4) it follows that $\tau_n^0|_{Q_1} = (2+\tau_{n-1}^0(1)) \cdot 1_{Q_1} + 2\omega_1 + \omega_2^+ + \omega_2^-$. By Lemma 3.8 $\tau_n^i(g) - \tau_n^0(g) = 1$ for any $g \in Q_1$. Hence $\tau_n^i|_{Q_1} = (2+\tau_{n-1}^i(1)) \cdot 1_Q + 2\omega_1 + \omega_2^+ + \omega_2^-$. By Corollary 4.2, Proposition 5.7, and Theorem 5.11, ω_2^+ lifts to the P_1-character $\boldsymbol{A}_+$, and ω_2^- lifts to the P_1-character $\boldsymbol{A}_-$. Since $\boldsymbol{C}\delta_{\pm i}|_{Q_1} = \omega_1$, $2\omega_1$ lifts to the

P_1-character $\boldsymbol{C}(\delta_i+\delta_{-i})$ by the result of 2). Clearly, $\tau_n^{q-1-i}=\tau_n^i=\mathrm{Ind}_{P_1}^G(\delta_i)$, whence $\tau_n^i|_{P_1}$ contains δ_i and δ_{-i} by Frobenius' reciprocity. It also follows from 4) that

$$(\boldsymbol{C}(\delta_i+\delta_{-i})+\boldsymbol{A}_-+\boldsymbol{A}_+)|_{L_1'}=2q\cdot 1_{L_1'}+(q+1)(2\sum_{k=1}^{(q-2)/2}\tau_{n-1}^k+\rho_{n-1}^1+\rho_{n-1}^2).$$

Together with Proposition 7.7, this implies that the constituent $(2+\tau_{n-1}^i(1))\cdot 1_{Q_1}$ of $\tau_n^i|_{Q_1}$ yields the L_1'-character $2\cdot 1_{L_1'}+\tau_{n-1}^i$. We have shown that

$$\tau_n^i|_{P_1}=\delta_i+\delta_{-i}+\tau_{n-1}^i\otimes\gamma+\boldsymbol{C}(\delta_i+\delta_{-i})+\boldsymbol{A}_++\boldsymbol{A}_-$$

for some linear character γ of T_1.

6) It remains to identify the T_1-characters α, β, and γ. Consider the semisimple elements $\boldsymbol{c}_j$ with $1\leq j\leq q/2-1$. Since $\boldsymbol{c}_j\in L_1\setminus L_1'$ and $Stab_{L_1}(\lambda)<L_1'$ for any $\lambda\in\mathcal{O}_2^\varepsilon$, $\boldsymbol{A}_+(\boldsymbol{c}_j)=\boldsymbol{A}_-(\boldsymbol{c}_j)=0$. Also, $\boldsymbol{c}_j$ fixes no $\lambda\in\mathcal{O}_1$, whence $(\boldsymbol{C}\delta_k)(\boldsymbol{c}_j)=0$ for any k. Thus the result of 5) implies that $\tau_n^i(\boldsymbol{c}_j)=\tilde{\delta}^{ij}+\tilde{\delta}^{-ij}+\tau_{n-1}^i(1)\cdot\gamma(\boldsymbol{c}_j)$. But $\tau_n^i(\boldsymbol{c}_j)=\tilde{\delta}^{ij}+\tilde{\delta}^{-ij}+\tau_{n-1}^i(1)$ by Corollary 7.8. Consequently, $\gamma(\boldsymbol{c}_j)=1$ for any j, i.e. $\gamma=1_{T_1}$. In the same fashion one can show that $\alpha=\beta=1_{T_1}$. $\square$

Corollary 7.10. *Let $G=Sp_{2n}(q)$ with $n\geq 2$. Write $q-1=\ell^a r$ with $(r,\ell)=1$.*

(i) *Let φ be any nontrivial composition factor of $\hat{\rho}_n^1$, $\hat{\rho}_n^2$, or $\hat{\tau}_n^i$. Then φ is of form $\rho-a\cdot 1_G$, where $\rho\in\{\hat{\rho}_n^1,\hat{\rho}_n^2,\hat{\tau}_n^j\mid 1\leq j\leq(r-1)/2\}$ and $a=0$ or 1. Moreover, $a=1$ exactly when either $\ell|(q^n-1)/(q-1)$ and $\rho=\hat{\rho}_n^1$, or $\ell|(q^n+1)$ and $\rho=\hat{\rho}_n^2$.*

(ii) *The Brauer characters $\hat{\rho}_n^1$, $\hat{\rho}_n^2$, $\hat{\tau}_n^j$ with $1\leq j\leq(r-1)/2$, each have a unique nontrivial irreducible constituent, and these $(r+3)/2$ constituents are pairwise distinct. This statement also holds for $n=1$ if we remove $\hat{\rho}_1^1$ from this list.*

Proof. (i) The statement for $n=2$, resp. for $(n,q)=(3,2)$ and $(4,2)$, can be checked directly using [Wh1], resp. [JLPW]. We will proceed by induction on n and assume $n\geq 3$ and $(n,q)\neq(3,2)$, $(4,2)$.

1) Arguing as in the proof of [DT, Theorem 7.2] one can show that

$$\hat{\tau}_n^i-\hat{\tau}_n^j=-\delta_{i,0}+\delta_{j,0},$$

whenever $0\leq i,j\leq q-2$ and $i\equiv j(\mathrm{mod}\, r)$. Together with (2) and (3), it implies that any nontrivial composition factor of $\hat{\tau}_n^i$ is a composition factor of $\hat{\rho}_n^1$, $\hat{\rho}_n^2$, or $\hat{\tau}_n^j$ with $1\leq j\leq(r-1)/2$.

2) Consider the semisimple character ψ_j labelled by $s_j\in G$ conjugate to $\mathrm{diag}(\delta^j,\delta^{-j},I_{2n-2})$ with $1\leq j\leq(r-1)/2$. Since the ℓ'-part s_j' of s_j is nontrivial semisimple and $\psi_j(1)=(q^{2n}-1)/(q-1)=(G:C_G(s_j'))_{2'}$, [HM, Proposition 1] implies that $\widehat{\psi}_j$ is irreducible. Moreover, these $(r-1)/2$ characters ψ_j belong to $(r-1)/2$ distinct nonunipotent ℓ-blocks, since the s_j' are pairwise nonconjugate and nontrivial. On the other hand, by Corollary 6.3, $\{\psi_j\mid 1\leq j\leq(r-1)/2\}\subseteq\{\tau_n^i\mid 1\leq i\leq q/2-1\}$. The result of 1) shows that there are at most $(r-1)/2$ distinct irreducible Brauer characters of degree $(q^{2n}-1)/(q-1)$ that can arise as composition factors of $\hat{\tau}_n^i$. We conclude that the Brauer characters $\hat{\tau}_n^j$ with $1\leq j\leq(r-1)/2$ are irreducible and pairwise distinct.

3) Notice that $1_G+\rho_n^1+\rho_n^2$ is the rank 3 permutation character for G, whence the statement that $\rho-a\cdot 1_G\in\mathrm{IBr}_\ell(G)$ for a suitable $a=0,1$ when $\rho\in\{\hat{\rho}_n^1,\hat{\rho}_n^2\}$ follows from [Li]. Furthermore, it is shown in [ST] that $\hat{\rho}_n^1$ is reducible if and only

if $\ell|(q^n-1)/(q-1)$, and $\hat{\rho}_n^2$ is reducible if and only if $\ell|(q^n+1)$. Thus (i) has been established.

(ii) follows from (i) if $n \geq 2$. The statement for $n=1$ is obvious. $\Box$

In view of Corollaries 7.5 and 7.10, define

$$\mathfrak{W}_+ := \{1_G, \hat{\rho}_n^1, \hat{\rho}_n^2, \hat{\tau}_n^j \mid 1 \leq j \leq (r-1)/2\},$$

where $r = (q-1)_{\ell'}$, and

$$\mathfrak{W}_- := \{1_G, \widehat{\alpha}_n, \widehat{\beta}_n, \hat{\zeta}_n^j \mid 1 \leq j \leq (r'-1)/2\},$$

where $r' = (q+1)_{\ell'}$. By Corollary 7.10, any linear-Weil (Brauer) character is a $\mathbb{Z}$-combination of characters from $\mathfrak{W}_+$. Similarly, any unitary-Weil (Brauer) character is a $\mathbb{Z}$-combination of characters from $\mathfrak{W}_-$ by Corollary 7.5.

Lemma 7.11. *Let $G = Sp_{2n}(q)$.*

(i) *Let $n \geq 1$ and let φ be a $\mathbb{Z}$-combination of trivial and linear-Weil Brauer characters. Suppose that $\varphi|_{P_1} = 0$. Then $\varphi = 0$.*

(ii) *Let $n \geq 2$ and let φ be a $\mathbb{Z}$-combination of trivial and unitary-Weil Brauer characters. Suppose that $\varphi|_{P_1} = 0$. Then $\varphi = 0$.*

Proof. (i) Write $(q-1) = \ell^k r$ with $(r,\ell) = 1$. Then $\varphi = a\hat{\rho}_n^1 + b\hat{\rho}_n^2 + \sum_{i=1}^{(r-1)/2} c_i\hat{\tau}_n^i + d\cdot 1_G$ for some $a, b, c_i, d \in \mathbb{Z}$. The case $n=1$ can be checked directly using character table, so we assume that $n \geq 2$. By assumption, $\varphi(g) = 0$ for $g = 1$, $\boldsymbol{t}$, $\boldsymbol{s}$, $\boldsymbol{d}$, and $\boldsymbol{c}_0^j$ with $1 \leq j \leq (r-1)/2$, where $\boldsymbol{c}_0 := \boldsymbol{c}_{\ell^k}$. The conditions $\varphi(1) = \varphi(\boldsymbol{t}) = \varphi(\boldsymbol{s}) = \varphi(\boldsymbol{d}) = 0$ imply by Corollary 7.8 that $a = b = d = 0$ and

$$\sum_{i=1}^{(r-1)/2} c_i = 0\ . \tag{21}$$

Let $\tilde{\delta}_0 := \delta^{\ell^k}$. By Corollary 7.8 and (21),

$$0 = \varphi(\boldsymbol{c}_0^j) = \sum_{i=1}^{(r-1)/2} c_i\left(\frac{q^{2n-2}-1}{q-1} + \tilde{\delta}_0^{ij} + \tilde{\delta}_0^{-ij}\right) = \sum_{i=1}^{(r-1)/2} c_i(\tilde{\delta}_0^{ij} + \tilde{\delta}_0^{-ij}). \tag{22}$$

Let $f(x) := \sum_{i=1}^{(r-1)/2} c_i(x^i + x^{r-i}) \in \mathbb{Z}[x]$. It follows from (21) and (22) that $f(\tilde{\delta}_0^j) = 0$ for $j = 0, 1, \ldots, r-1$. But $\deg(f) \leq r-1$, so $f \equiv 0$, i.e. $c_i = 0$ for all i.

(ii) Write $(q+1) = \ell^k r$ with $(r,\ell) = 1$. Then $\varphi = a\widehat{\alpha}_n + b\widehat{\beta}_n + \sum_{i=1}^{(r-1)/2} c_i\hat{\zeta}_n^i + d\cdot 1_G$ for some $a, b, c_i, d \in \mathbb{Z}$. The case $n=2$, resp. $(n,q) = (3,2)$ or $(4,2)$, can be checked directly using the character table of G [E], resp. [JLPW], so we assume that $n \geq 3$ and $(n,q) \neq (3,2)$, $(4,2)$. By Proposition 7.4, the Q_1-fixed point part of φ yields the (virtual) L_1'-character

$$\psi := d\cdot 1_{L_1'} + a\widehat{\alpha}_{n-1} + b\widehat{\beta}_{n-1} + \sum_{i=1}^{(r-1)/2} c_i\hat{\zeta}_{n-1}^i.$$

By our assumption, $\psi = 0$. This implies by Corollary 7.5(ii) that $a = b = c_i = 0$ and $d = 0$. $\Box$

8. Restrictions to P_1 and H_d

In this section we will show that any $\mathbb{F}G$-module with property $(\mathcal{W}_2^{\varepsilon})$ agrees with a formal sum of Weil modules when restricted to P_1 and H_d.

Proposition 8.1. *Let $G = Sp_{2n}(q)$ with $n \geq 3$, $(n,q) \neq (3,2)$, $(4,2)$, and let V be an irreducible $\mathbb{F}G$-module. Assume that V satisfies $(\mathcal{W}_2^-)$. Then there is a formal sum W of trivial and (irreducible) unitary-Weil modules of G such that $V|_{P_1} = W|_{P_1}$.*

Proof. Write $q+1 = \ell^k r$ with $(r,\ell) = 1$. Let φ be the Brauer character of V.

1) Without loss we may assume that V is nontrivial. By Proposition 5.7 and Theorem 5.11, V satisfies the conclusion of Theorem 5.11(i). It follows that

$$V|_{P_1} = C_V(Q_1) \oplus n_+\widehat{\boldsymbol{B}}_+ \oplus n_-\widehat{\boldsymbol{B}}_- \tag{23}$$

for some nonnegative integers n_+, n_- (in the notation of §7). According to (16) and (17), all nontrivial composition factors of $\widehat{\boldsymbol{B}}_+|_{L_1'}$ and $\widehat{\boldsymbol{B}}_-|_{L_1'}$ are unitary-Weil modules of L_1'. Hence the same holds for $C_V(Q_1)|_{L_1'}$ by Lemma 2.1. By Corollary 7.5 applied to L_1',

$$C_V(Q_1)|_{L_1'} = a\widehat{\alpha}_{n-1} + b\widehat{\beta}_{n-1} + \sum_{i=1}^{(r-1)/2} c_i\hat{\zeta}_{n-1}^i + d \cdot 1_{L_1'} \tag{24}$$

for some $a, b, c_i, d \in \mathbb{Z}$. Notice that $a, b, c_i \geq 0$.

2) Here we compute $\varphi(\boldsymbol{t})$ and $\varphi(\boldsymbol{t}')$ for transvections $\boldsymbol{t} \in Z_1$ and $\boldsymbol{t}' \in L_1'$. By Proposition 7.4 and Corollary 7.3,

$$\boldsymbol{B}_-(\boldsymbol{t}) = \alpha_n(\boldsymbol{t}) - \alpha_{n-1}(1) = \frac{-q^{2n-2}+q^{n-1}}{2},$$
$$\boldsymbol{B}_-(\boldsymbol{t}') = \alpha_n(\boldsymbol{t}') - \alpha_{n-1}(\boldsymbol{t}') = \frac{-q^{2n-2}+q^{2n-3}}{2}\ .$$

Similarly,

$$\boldsymbol{B}_+(\boldsymbol{t}) = (-q^{2n-2} - q^{n-1})/2\ ,\ \boldsymbol{B}_+(\boldsymbol{t}') = (-q^{2n-2} + q^{2n-3})/2\ .$$

Now we have

$$\begin{aligned}\varphi(\boldsymbol{t}) &= a\frac{(q^{n-1}-1)(q^{n-1}-q)}{2(q+1)} + b\frac{(q^{n-1}+1)(q^{n-1}+q)}{2(q+1)} + \sum_i c_i\frac{q^{2n-2}-1}{q+1} + d \\ &\quad + n_+\frac{-q^{2n-2}-q^{n-1}}{2} + n_-\frac{-q^{2n-2}+q^{n-1}}{2}, \\ \varphi(\boldsymbol{t}') &= (a+b)\frac{-q^{2n-3}+q}{2(q+1)} + \sum_i c_i\frac{-q^{2n-3}-1}{q+1} + d + (n_+ + n_-)\frac{-q^{2n-2}+q^{2n-3}}{2}.\end{aligned}$$

But $\varphi(\boldsymbol{t}) = \varphi(\boldsymbol{t}')$, hence

$$(a + \sum_i c_i - n_-)(q^{n-2}-1) + (b + \sum_i c_i - n_+)(q^{n-2}+1) = 0. \tag{25}$$

3) Here we compute $\varphi(\boldsymbol{d})$ and $\varphi(\boldsymbol{d}')$ for double transvections $\boldsymbol{d} = \boldsymbol{t}\boldsymbol{t}'$ (with $\boldsymbol{t} \in Z_1$ and $\boldsymbol{t}' \in L_1'$ as in 2)) and $\boldsymbol{d}' \in L_1'$. By Proposition 7.4 and Corollary 7.3,

$$\boldsymbol{B}_-(\boldsymbol{d}) = \alpha_n(\boldsymbol{d}) - \alpha_{n-1}(\boldsymbol{t}') = \frac{q^{2n-3}}{2},\ \boldsymbol{B}_-(\boldsymbol{d}') = \alpha_n(\boldsymbol{d}') - \alpha_{n-1}(\boldsymbol{d}') = \frac{q^{2n-3}-q^{2n-4}}{2}\ .$$

Similarly,

$$\boldsymbol{B}_+(\boldsymbol{t}) = q^{2n-3}/2\ ,\ \boldsymbol{B}_+(\boldsymbol{d}') = (q^{2n-3} - q^{2n-4})/2\ .$$

Now we have

$$\varphi(\boldsymbol{d}) = (a+b)\frac{-q^{2n-3}+q}{2(q+1)} + \sum_i c_i \frac{-q^{2n-3}-1}{q+1} + d + (n_+ + n_-)q^{2n-3}/2,$$

$$\varphi(\boldsymbol{d}') = (a+b)\frac{q^{2n-4}+q}{2(q+1)} + \sum_i c_i \frac{q^{2n-4}-1}{q+1} + d + (n_+ + n_-)\frac{q^{2n-3}-q^{2n-4}}{2}.$$

But $\varphi(\boldsymbol{d}) = \varphi(\boldsymbol{d}')$, hence

$$(a + \sum_i c_i - n_-) + (b + \sum_i c_i - n_+) = 0. \tag{26}$$

4) From (25) and (26) it follows that

$$n_- = a + \sum_i c_i,\ n_+ = b + \sum_i c_i. \tag{27}$$

Now we define W to be the (virtual) $\mathbb{F}G$-module with Brauer character $a\widehat{\alpha}_n + b\widehat{\beta}_n + \sum_{i=1}^{(r-1)/2} c_i\hat{\zeta}_n^i + d\cdot 1_G$. By Proposition 7.4,

(28)

$$W|_{P_1} = (a\widehat{\alpha}_{n-1} + b\widehat{\beta}_{n-1} + \sum_i c_i\hat{\zeta}_{n-1}^i)\otimes 1_{T_1} + (a + \sum_i c_i)\widehat{\boldsymbol{B}}_- + (b + \sum_i c_i)\widehat{\boldsymbol{B}}_+ + d\cdot 1_G.$$

Then equalities (23), (24), (27), and (28) imply that $V|_{P_1'} = W|_{P_1'}$ and moreover $[V, Q_1] = [W, Q_1]$ as P_1-modules.

5) Next we consider a conjugate T_2 of T_1 that is contained in L_1'. Since $T_2 < P_1'$, $V = W$ as T_2-modules. But T_1 and T_2 are conjugate, hence $V = W$ as T_1-modules. Now $T_1 < P_1$, whence $[V, Q_1] = [W, Q_1]$ as T_1-modules. It follows that $C_V(Q_1) = C_W(Q_1)$ as T_1-modules. But T_1 acts trivially on $C_W(Q_1)$ by (28), so T_1 also acts trivially on $C_V(Q_1)$.

Define $e := 1$ if $\ell|(q+1)$ and 0 otherwise. By Corollary 7.5, $\widehat{\beta}_{n-1} = \phi_{n-1} + e\cdot 1_{L_1'}$ where $\phi_{n-1} \in \mathrm{IBr}_\ell(L_1')$. Thus instead of (24) we have

$$C_V(Q_1)|_{L_1'} = a\widehat{\alpha}_{n-1} + b\phi_{n-1} + \sum_{i=1}^{(r-1)/2} c_i\hat{\zeta}_{n-1}^i + (d+e)\cdot 1_{L_1'}, \tag{29}$$

where $d+e \geq 0$ since it is the multiplicity of $1_{L_1'}$ in $C_V(Q_1)|_{L_1'}$. We have mentioned above that $a, b, c_i \geq 0$. Hence the trivial action of T_1 on $C_V(Q_1)$ (and (29)) implies that

$$C_V(Q_1)|_{L_1} = a\widehat{\alpha}_{n-1}\otimes 1_{T_1} + b\phi_{n-1}\otimes 1_{T_1} + \sum_{i=1}^{(r-1)/2} c_i\hat{\zeta}_{n-1}^i\otimes 1_{T_1} + (d+e)\cdot 1_{L_1},$$

and so

$$C_V(Q_1)|_{L_1} = a\widehat{\alpha}_{n-1}\otimes 1_{T_1} + b\widehat{\beta}_{n-1}\otimes 1_{T_1} + \sum_{i=1}^{(r-1)/2} c_i\hat{\zeta}_{n-1}^i\otimes 1_{T_1} + d\cdot 1_{L_1}. \tag{30}$$

It follows from (23), (30), and (28) that $V = W$ as P_1-modules. $\square$

Proposition 8.2. *Let $G = Sp_{2n}(q)$ with $n \geq 3$, $(n,q) \neq (3,2)$, $(4,2)$ and let V be an irreducible $\mathbb{F}G$-module. Assume that V satisfies $(\mathcal{W}_2^+)$. Then there is a formal sum W of trivial and (irreducible) linear-Weil modules of G such that $V|_{P_1'} = W|_{P_1'}$.*

Proof. Write $q-1 = \ell^k r$ with $(r,\ell)=1$. Let φ be the Brauer character of V.

1) Without loss we may assume that V is nontrivial. By Proposition 5.7 and Theorem 5.11, V satisfies the conclusion of Theorem 5.11(ii). It follows that

$$V|_{P_1'} = C_V(Q_1) \oplus (m\widehat{\boldsymbol{C}} + n_+\widehat{\boldsymbol{A}}_+ + n_-\widehat{\boldsymbol{A}}_-)|_{P_1'} \tag{31}$$

for some nonnegative integers m, n_+, n_- (in the notation of §7). According to Proposition 7.9, all nontrivial composition factors of $\widehat{\boldsymbol{C}}|_{L_1'}$, $\widehat{\boldsymbol{A}}_+|_{L_1'}$, and $\widehat{\boldsymbol{A}}_-|_{L_1'}$ are contained in $\hat{\rho}_n^1|_{L_1'}$ and $\hat{\rho}_n^2|_{L_1'}$, and therefore they are linear-Weil modules of L_1' by Proposition 7.7. The same also holds for $C_V(Q_1)|_{L_1'}$ by Lemma 2.1. By Corollary 7.10 applied to L_1',

$$C_V(Q_1)|_{L_1'} = a\hat{\rho}_{n-1}^1 + b\hat{\rho}_{n-1}^2 + \sum_{i=1}^{(r-1)/2} c_i\hat{\tau}_{n-1}^i + d\cdot 1_{L_1'} \tag{32}$$

for some $a, b, c_i, d \in \mathbb{Z}$. Notice that $a, b, c_i \geq 0$. Define W to be the (virtual) $\mathbb{F}G$-module with Brauer character

$$a\hat{\rho}_n^1 + b\hat{\rho}_n^2 + \sum_{i=1}^{(r-1)/2} c_i\hat{\tau}_n^i + (d-a-b-2\sum_{i=1}^{(r-1)/2} c_i)\cdot 1_G.$$

Then Proposition 7.9 and (31), (32) imply that

$$\psi|_{P_1'} = x\widehat{\boldsymbol{C}} + y_-\widehat{\boldsymbol{A}}_- + y_+\widehat{\boldsymbol{A}}_+, \tag{33}$$

where ψ is the Brauer character of the (virtual) $\mathbb{F}G$-module $W-V$, $c := \sum_{i=1}^{(r-1)/2} c_i$, $x := a+b+2c-m$, $y_- := a+c-n_-$, and $y_+ := b+c-n_+$.

2) Here we compute $\psi(\boldsymbol{t})$ and $\psi(\boldsymbol{t}')$ for transvections $\boldsymbol{t} \in Z_1$ and $\boldsymbol{t}' \in L_1'$. Notice that for any $h \in L_1'$, $\boldsymbol{C}(h)+1$ is just the number of h-fixed points on $\langle e_2, \ldots, e_n, f_2, \ldots, f_n\rangle_{\mathbb{F}_q}$; in particular, $\boldsymbol{C}(\boldsymbol{t}') = q^{2n-3}-1$. Furthermore, any $\lambda \in \mathcal{O}_1$ is trivial on Z_1, whence $\boldsymbol{C}(\boldsymbol{t}) = \boldsymbol{C}(1)$. By Proposition 7.9 and Corollary 7.8,

$$\begin{aligned} \boldsymbol{A}_-(\boldsymbol{t}) &= \rho_n^1(\boldsymbol{t}) - 1 - \rho_{n-1}^1(1) - \boldsymbol{C}(1) = (-q^{2n-2}+q^{n-1})/2, \\ \boldsymbol{A}_-(\boldsymbol{t}') &= \rho_n^1(\boldsymbol{t}') - 1 - \rho_{n-1}^1(\boldsymbol{t}') - \boldsymbol{C}(\boldsymbol{t}') = (q^{2n-2}-q^{2n-3})/2\ . \end{aligned}$$

Similarly,

$$\boldsymbol{A}_+(\boldsymbol{t}) = (-q^{2n-2}-q^{n-1})/2\ ,\ \boldsymbol{A}_+(\boldsymbol{t}') = (q^{2n-2}-q^{2n-3})/2\ .$$

Now we have

$$\begin{aligned} \psi(\boldsymbol{t}) &= x(q^{2n-2}-1) + y_-(-q^{2n-2}+q^{n-1})/2 + y_+(-q^{2n-2}-q^{n-1})/2, \\ \psi(\boldsymbol{t}') &= x(q^{2n-3}-1) + y_-(q^{2n-2}-q^{2n-3})/2 + y_+(q^{2n-2}-q^{2n-3})/2. \end{aligned}$$

But $\psi(\boldsymbol{t}) = \psi(\boldsymbol{t}')$, hence

$$2x(q^{n-1}-q^{n-2}) = y_-(2q^{n-1}-q^{n-2}-1) + y_+(2q^{n-1}-q^{n-2}+1). \tag{34}$$

3) Next we compute $\psi(\boldsymbol{d})$, $\psi(\boldsymbol{d}')$, and $\psi(\boldsymbol{d}'')$ for double transvections $\boldsymbol{d} = \boldsymbol{t}\boldsymbol{t}'$ (where $\boldsymbol{t}$ and $\boldsymbol{t}'$ are as in 2)), $\boldsymbol{d}' \in L_1'$, and $\boldsymbol{d}'' = \mathrm{diag}\left(\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, I_{2n-4}\right)$ (in

the basis $(e_1, e_2, f_1, f_2, e_3, \dots, e_n, f_3, \dots, f_n)$). Observe that $\boldsymbol{d}'' \in Q_1 \setminus Z_1$, whence $\boldsymbol{C}(\boldsymbol{d}'') = -1$. Furthermore, $\boldsymbol{C}(\boldsymbol{d}) = \boldsymbol{C}(\boldsymbol{t}') = q^{2n-3} - 1$, and $\boldsymbol{C}(\boldsymbol{d}') = q^{2n-4} - 1$. By Proposition 7.9 and Corollary 7.8,

$$\begin{aligned} \boldsymbol{A}_-(\boldsymbol{d}) = \ & \rho^1_n(\boldsymbol{d}) - 1 - \rho^1_{n-1}(\boldsymbol{t}') - \boldsymbol{C}(\boldsymbol{d}) = -q^{2n-3}/2, \\ \boldsymbol{A}_-(\boldsymbol{d}') = \ & \rho^1_n(\boldsymbol{d}') - 1 - \rho^1_{n-1}(\boldsymbol{d}') - \boldsymbol{C}(\boldsymbol{d}') = (q^{2n-3} - q^{2n-4})/2, \\ \boldsymbol{A}_-(\boldsymbol{d}'') = \ & \rho^1_n(\boldsymbol{d}'') - 1 - \rho^1_{n-1}(1) - \boldsymbol{C}(\boldsymbol{d}'') = q^{n-1}/2, \end{aligned}$$

Similarly,

$$\boldsymbol{A}_+(\boldsymbol{d}) = -q^{2n-3}/2\ ,\ \boldsymbol{A}_+(\boldsymbol{d}') = (q^{2n-3} - q^{2n-4})/2\ ,\ \boldsymbol{A}_+(\boldsymbol{d}'') = -q^{n-1}/2\ .$$

Now we obtain

$$\begin{aligned} \psi(\boldsymbol{d}) &= x(q^{2n-3} - 1) - (y_- + y_+)q^{2n-3}/2\ , \\ \psi(\boldsymbol{d}') &= x(q^{2n-4} - 1) + (y_- + y_+)(q^{2n-3} - q^{2n-4})/2\ , \\ \psi(\boldsymbol{d}'') &= -x + (y_- - y_+)q^{n-1}/2\ . \end{aligned}$$

But $\psi(\boldsymbol{d}) = \psi(\boldsymbol{d}') = \psi(\boldsymbol{d}'')$, hence

(35)

$$x = \frac{(y_- + y_+)(2q-1)}{2(q-1)},\ 2xq^{n-3} + y_-(q^{n-2} - q^{n-3} - 1) + y_+(q^{n-2} - q^{n-3} + 1) = 0.$$

The equations (34) and (35) yield $x = y_+ = y_- = 0$. Thus $\psi|_{P'_1} = 0$ by (33), i.e. $V = W$ as P'_1-modules. □

Now we can prove an analogue of Proposition 8.1 for linear-Weil characters.

Proposition 8.3. *Let $G = Sp_{2n}(q)$ with $n \geq 3$, $(n, q) \neq (3, 2)$, $(4, 2)$ and let V be an irreducible $\mathbb{F}G$-module. Assume that V satisfies $(\mathcal{W}_2^+)$. Then there is a formal sum W of trivial and (irreducible) linear-Weil modules of G such that $V|_{P_1} = W|_{P_1}$.*

Proof. 1) By Proposition 8.2, there is a formal sum W of trivial and (irreducible) linear-Weil modules of G such that $V|_{P'_1} = W|_{P'_1}$. We will keep all the notation used in the proof of Proposition 8.2. For any $\lambda \in \mathrm{IBr}_\ell(Q_1)$ let V_λ denote the λ-homogeneous component of V. Clearly, $V = C_V(Q_1) \oplus V_1 \oplus V_2^+ \oplus V_2^-$ as P_1-modules, where $V_1 = \sum_{\lambda \in \mathcal{O}_1} V_\lambda$, $V_2^\varepsilon = \sum_{\lambda \in \mathcal{O}_2^\varepsilon} V_\lambda$. Similarly, $W = C_W(Q_1) \oplus W_1 \oplus W_2^+ \oplus W_2^-$. The proof of Proposition 8.2 also shows that $V_2^\varepsilon = W_2^\varepsilon$ as P_1-modules for $\varepsilon = \pm$. It remains to prove that $V_1 = W_1$ and $C_V(Q_1) = C_W(Q_1)$ as P_1-modules. Now (32) implies

$$C_V(Q_1)|_{L_1} = \hat{\rho}^1_{n-1} \otimes \alpha + \hat{\rho}^2_{n-1} \otimes \beta + \sum_{i=1}^{(r-1)/2} \hat{\tau}^i_{n-1} \otimes \mu_i + 1_{L'_1} \otimes \eta, \tag{36}$$

where α, resp. β, μ_i, are Brauer characters of T_1 of degree a, resp. b, c_i; furthermore, η is a virtual Brauer character of T_1 of degree d. On the other hand, the choice of W and Proposition 7.9 yield

(37)

$$\begin{aligned} C_W(Q_1)|_{L_1} = \ & \Big(a\hat{\rho}^1_{n-1} + b\hat{\rho}^2_{n-1} + \textstyle\sum_{i=1}^{(r-1)/2} c_i\hat{\tau}^i_{n-1} + (d - 2\sum_{i=1}^{(r-1)/2} c_i)1_{L'_1}\Big) \\ & \otimes 1_{T_1} + \textstyle\sum_{i=1}^{(r-1)/2} c_i 1_{L'_1} \otimes (\delta_i + \delta_{-i}). \end{aligned}$$

One can also show that there are Brauer characters γ and γ' of P_1 that are trivial on P'_1 and of same degree m such that

$$V_1|_{P_1} = \boldsymbol{C}\gamma,\ W_1|_{P_1} = \boldsymbol{C}\gamma'. \tag{38}$$

Observe that

$$\boldsymbol{C}\gamma = \mathrm{Ind}_{K_\lambda}^{P_1}(\hat{\lambda}\cdot\gamma|_{K_\lambda}),\ \boldsymbol{C}\gamma' = \mathrm{Ind}_{K_\lambda}^{P_1}(\hat{\lambda}\cdot\gamma'|_{K_\lambda}) \tag{39}$$

for $\lambda \in \mathcal{O}_1$.

2) Consider the following two subgroups of G:

$$R_1 = Stab_G(\langle e_1\rangle_{\mathbb{F}_q}, e_2, f_2),\ R_2 = Stab_G(\langle e_2\rangle_{\mathbb{F}_q}, e_1, f_1).$$

Since $R_2 < P_1'$, $V = W$ as R_2-modules. But R_1 and R_2 are conjugate, whence $V = W$ as R_1-modules. Since $R_1 < P_1$, we see that $C_V(Q_1)\oplus V_1 = C_W(Q_1)\oplus W_1$ as R_1-modules. Observe that $R_1 = \widetilde{Q}_1 : \widetilde{L}_1$ plays the role of the P_1'-subgroup in $Stab_G(e_2, f_2) \simeq Sp_{2n-2}(q)$, with $\widetilde{Q}_1 = [q^{2n-3}]$ and $\widetilde{L}_1 := Stab_{R_1}(\langle f_1\rangle_{\mathbb{F}_q}) \simeq Sp_{2n-4}(q)\times\mathbb{Z}_{q-1}$.

Fix the Q_1-character $\lambda := \lambda_B\ :\ [A, C] \mapsto (-1)^{\mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_2}({}^t bJ_{n-1}A)}$, where $b = e_3$ (here we identify $\mathbb{F}_q^{2n-2}$ with $\langle e_2, \ldots, e_n, f_2, \ldots, f_n\rangle_{\mathbb{F}_q}$). Let $\pi := \lambda|_{\widetilde{Q}_1}$. Define q^2 distinct Q_1-characters $\lambda_{x,y} := \lambda_{b_{x,y}}$, where $b_{x,y} := xe_2 + e_3 + yf_2$ for $x, y \in \mathbb{F}_q$. Then these $\lambda_{x,y}$ are exactly the Q_1-characters from $\mathcal{O}_1$ that are equal to π when restricted to $\widetilde{Q}_1$. Furthermore, $R_\pi := Stab_{R_1}(\pi) = \widetilde{Q}_1 : (J : T')$, where $J := \{g \in \widetilde{L}_1 \mid g(b) = b\}$ and

$$T' := \{\boldsymbol{z} := \mathrm{diag}(z, 1, z, I_{n-3}, 1, z^{-1}, I_{n-3}, z^{-1}) \mid z \in \mathbb{F}_q^\bullet\}$$

(in the basis $(e_1, e_2, \ldots, e_n, f_2, \ldots, f_n, f_1)$). By Lemma 2.3, the π-homogeneous components $\widetilde{V}_\pi$ (of $(C_V(Q_1)\oplus V_1)|_{\widetilde{Q}_1}$) and $\widetilde{W}_\pi$ (of $(C_W(Q_1)\oplus W_1)|_{\widetilde{Q}_1}$) are equal as R_π-modules.

It is not difficult to see that $\widetilde{V}_\pi = \sum_{x,y\in\mathbb{F}_q} V_{\lambda_{x,y}}$ (and $\widetilde{W}_\pi = \sum_{x,y\in\mathbb{F}_q} W_{\lambda_{x,y}}$) affords the Q_1-character $m\sum_{x,y\in\mathbb{F}_q}\lambda_{x,y}$, and $\boldsymbol{z} \in T'$ maps $\lambda_{x,y}$ to $\lambda_{z^{-1}x,z^{-1}y}$. Let $\boldsymbol{z} \in T'$ be any nontrivial ℓ'-element. Then $\boldsymbol{z}$ acts regularly on $\{\lambda_{x,y} \mid (0,0) \neq (x,y) \in \mathbb{F}_q^2\}$. Hence the trace of $\boldsymbol{z}$ acting on $\widetilde{V}_\pi$ equals the trace of $\boldsymbol{z}$ acting on V_λ. Recall that $\boldsymbol{z} \in I_\lambda$ and $\mathrm{Ker}(\hat{\lambda}) > I_\lambda$. Therefore, (38) and (39) imply that the trace of $\boldsymbol{z}$ acting on V_λ is $\gamma(\boldsymbol{z})$. Thus the trace of $\boldsymbol{z}$ acting on $\widetilde{V}_\pi$ equals $\gamma(\boldsymbol{z})$. Similarly, the trace of $\boldsymbol{z}$ acting on $\widetilde{W}_\pi$ equals $\gamma'(\boldsymbol{z})$. Since $\widetilde{V}_\pi = \widetilde{W}_\pi$ as R_π-modules, it follows that $\gamma(\boldsymbol{z}) = \gamma'(\boldsymbol{z})$, for any ℓ'-element $1 \neq \boldsymbol{z} \in T'$. But $\gamma(1) = \gamma'(1) = m$. Hence $\gamma|_{T'} = \gamma'|_{T'}$ (as Brauer characters). Finally, observe that $P_1 = P_1'T'$, and γ and γ' are actually P_1/P_1'-characters. Consequently, $\gamma = \gamma'$, and so $V_1 = W_1$ as P_1-modules, because of (38).

3) The results of 2) imply that $C_V(Q_1) = C_W(Q_1)$ as R_1-modules. Since $\widetilde{Q}_1 < Q_1$, we can also say that $C_V(Q_1) = C_W(Q_1)$ as modules over $\widetilde{L}_1 = \widetilde{L}_1' \times T_1 \simeq Sp_{2n-4}(q)\times\mathbb{Z}_{q-1}$. By Corollary 7.10 there is a unique nontrivial composition factor ρ of $\hat{\rho}_{n-2}^2$ (as an $\widetilde{L}_1'$-character). By Proposition 7.7, Corollary 7.10, and (36), the ρ-homogeneous component of $C_V(Q_1)|_{\widetilde{L}_1'}$ affords the T_1-character

$$(1 + (q+1)\frac{\ell^k-1}{2})\alpha + (q+1)(1 + \frac{\ell^k-1}{2})\beta + (q+1)\ell^k\sum_{i=1}^{(r-1)/2}\mu_i,$$

and the ρ-homogeneous component of $C_W(Q_1)|_{\widetilde{L}_1'}$ affords the T_1-character

$$\left((1 + (q+1)\frac{\ell^k-1}{2})a + (q+1)(1 + \frac{\ell^k-1}{2})b + (q+1)\ell^k\sum_{i=1}^{(r-1)/2}c_i\right)1_{T_1}.$$

These last two T_1-characters are equal, and $\alpha(1) = a \geq 0$, $\beta(1) = b \geq 0$, $\mu_i(1) = c_i \geq 0$. Hence

$$\alpha = a \cdot 1_{T_1},\ \beta = b \cdot 1_{T_1},\ \mu_i = c_i \cdot 1_{T_1}. \tag{40}$$

In this case, the equality of $\widetilde{L}_1$-modules $C_V(Q_1)$ and $C_W(Q_1)$ reduces to

$$\eta = (d - 2\sum_{i=1}^{(r-1)/2} c_i)1_{T_1} + \sum_{i=1}^{(r-1)/2} c_i(\delta_i + \delta_{-i}). \tag{41}$$

Now (36), (37), (40), and (41) imply that $C_V(Q_1) = C_W(Q_1)$ as P_1-modules, as desired. □

The main result of this section is the following

Theorem 8.4. *Let $G = Sp_{2n}(q)$ with $n \geq 3$, $(n, q) \neq (3, 2), (4, 2)$, and let V be an irreducible $\mathbb{F}G$-module.*

(i) *Assume that V satisfies $(\mathcal{W}_2^+)$. Then there is a formal sum W of trivial and (irreducible) linear-Weil modules of G such that $V = W$ as P_1-modules and also as H_d-modules for $1 \leq d \leq n/2$.*

(ii) *Assume that V satisfies $(\mathcal{W}_2^-)$. Then there is a formal sum W of trivial and (irreducible) unitary-Weil modules of G such that $V = W$ as P_1-modules and also as H_d-modules for $1 \leq d \leq n/2$.*

Proof. (i) By Proposition 8.3, there is a formal sum W of trivial and (irreducible) linear-Weil modules of G such that $V|_{P_1} = W|_{P_1}$. We need to show that $V = W$ as H_d-modules for $1 \leq d \leq n/2$. Recall that $H_d = A \times B$, where $A \simeq Sp_{2n-2d}(q)$ and $B \simeq Sp_{2d}(q)$. We may conjugate H_d so that $A \leq L_1' < P_1$. Then all composition factors of $W|_A$ are either trivial or linear-Weil by Lemma 3.7. The same is true for $V|_A$, since $V = W$ as P_1-modules and $A < P_1$. The same also holds for $W|_B$ and $V|_B$, since B is G-conjugate to a subgroup of L_1'. For the virtual module $U := W - V$, we may therefore write

$$U|_{H_d} = \sum_{j=1}^{(r+5)/2} M_j \otimes N_j,$$

where N_j runs over the set $\mathfrak{W}_+(B)$ consisting of 1_B and $(r+3)/2$ distinct irreducible linear-Weil modules of B, and each M_j is a formal sum of trivial and linear-Weil modules of A.

Consider the P_1-parabolic subgroup P_1^* of A. We may conjugate H_d so that $P_1^* \times B < P_1$. Since $U|_{P_1} = 0$, we have

$$0 = U|_{P_1^* \times B} = \sum_{j=1}^{(r+5)/2} (M_j|_{P_1^*}) \otimes N_j.$$

But the N_j are all distinct irreducible B-modules, hence the last equality implies that $M_j|_{P_1^*} = 0$ for each j. By Lemma 7.11 applied to B, $M_j = 0$ for all j. Consequently, $U = 0$ as H_d-modules, i.e. $V = W$ as H_d-modules.

(ii) Similar to (i). □

9. Restrictions to P_j, $2 \le j \le n-1$

In this section we consider the restriction to any parabolic subgroup P_j with $2 \le j \le n-1$. If V is a (virtual) $\mathbb{F}G$-module satisfying $(\mathcal{W}_2^{\varepsilon})$, then we can write

$$V|_{P_j} = C_V(Q_j) \oplus V_0 \oplus V_1 \oplus V_2. \tag{42}$$

Here Z_j acts trivially on V_0 and all Q_j-characters on V_0 are nontrivial. Furthermore, all Z_j-characters occuring in V_1, resp. in V_2, belong to $\mathcal{O}_1$, resp. to $\mathcal{O}_2^{\varepsilon}$. For any Z_j-character λ, let $K_\lambda := Stab_{P_j}(\lambda)$ and $I_\lambda := Stab_{L_j}(\lambda)$.

Throughout this section, we assume that $G = Sp_{2n}(q)$ with $n \ge 3$, $(n,q) \ne (3,2), (4,2)$, and that V is an irreducible $\mathbb{F}G$-module satisfying $\mathcal{W}_2^{\varepsilon}$ for some ε. By Theorem 8.4, there is a formal sum W of trivial and Weil modules of G such that $V = W$ as P_1-modules and also as H_d-modules for $1 \le d \le n/2$.

First we focus on the summand V_2 in the case $j = 2$. We keep the notation introduced in §3 before Lemma 3.1.

Lemma 9.1. *Let $\lambda = \lambda_B$ be a nontrivial linear character of Z_2 that belongs to $\mathcal{O}_2^{\varepsilon}$. Suppose that $x \in I_\lambda$ induces an element of odd order in $O(q_B)$. Then x acts trivially on the complete inverse image of $C_{Q_2/Z_2}(x)$ in $Q_2/\operatorname{Ker}(\lambda_B)$.*

Proof. Let $g = [A, C] \in Q_2$ be centralized by x modulo Z_2. We need to show that $h := xgx^{-1}g^{-1} \in \operatorname{Ker}(\lambda_B)$. Since x stabilizes λ_B, $x = \operatorname{diag}(X, Y, {}^tX^{-1})$ with $X \in O(q_B)$ and $Y \in Sp_{2n-4}(q)$. By assumption, X has odd order say m. Since $\operatorname{rank}(q_B) = 2$, $D := B + {}^tB = \begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix}$ for some $d \in \mathbb{F}_q^\bullet$. The condition $X \in O(q_B)$ implies that ${}^tXDX = D$ and ${}^tXBX - B \in \mathcal{H}_2^0(q)$, i.e. ${}^tXBX - B = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$ for some $b \in \mathbb{F}_q$, whence ${}^tXBX = B + bd^{-1}D$. It follows that $B = {}^tX^mBX^m = B + mbd^{-1}D$, i.e. $mbd^{-1}D = 0$. But m is odd, hence $b = 0$, and ${}^tXBX = B$.

Since $xgx^{-1} \equiv g (\operatorname{mod} Z_2)$, we have $h = [0, XC\,{}^tX + C]$. Observe that

$$\operatorname{Tr}(B(XC\,{}^tX + C)) = \operatorname{Tr}({}^tXBX \cdot C) + \operatorname{Tr}(BC) = \operatorname{Tr}(BC) + \operatorname{Tr}(BC) = 0.$$

Thus $\lambda_B(h) = 1$ as stated. □

Lemma 9.2. *Let $\lambda = \lambda_B$ be a nontrivial linear character of Z_2 that belongs to $\mathcal{O}_2^{\varepsilon}$. Then there is a unique irreducible ℓ-Brauer character μ of Q_2 of degree q^{2n-4} such that $\mu|_{Z_2} = \mu(1)\lambda$. Moreover μ extends to a Brauer character $\tilde{\mu}$ of K_λ. If $x \in I_\lambda$ is an ℓ'-element that induces an element of odd order in $O(q_B)$, then $\tilde{\mu}(x) \ne 0$.*

Proof. One can check that $(Z_2 : \operatorname{Ker}(\lambda_B)) = (Q_2' : (\operatorname{Ker}(\lambda_B) \cap Q_2')) = 2$ and moreover $Q_2/\operatorname{Ker}(\lambda_B)$ is an extraspecial 2-group of order $2q^{4n-8}$. Hence there exists a unique $\mu \in \operatorname{IBr}_\ell(Q_2)$ of degree q^{2n-4} such that $\mu|_{Z_2} = \mu(1)\lambda$.

Assume $\varepsilon = +$. Using (7) and Corollary 7.8, one can show that

$$\rho_n^1|_{Z_2} = (q^{2n-3} + q^{2n-4} + \frac{(q^{n-2}-q)(q^{n-2}+1)}{2(q-1)}) \cdot 1_{Z_2} + \frac{q^{2n-4} - q^{n-2}}{2}\,\omega_1 + q^{2n-4}\omega_2^+.$$

Thus the multiplicity of λ in $\rho_n^1|_{Z_2}$ is q^{2n-4}. Let A be an $\mathbb{F}G$-module that affords the unique nontrivial irreducible constituent of $\hat{\rho}_n^1$ (cf. Corollary 7.10). Then A_λ has dimension q^{2n-4}. Clearly, Z_2 acts on A_λ with character $\mu(1)\lambda$, and both Q_2 and K_λ act on A_λ. Thus μ extends to the K_λ-character $\tilde{\mu}$ of A_λ. Let $\nu \in \operatorname{Irr}(Q_2)$ be such that $\nu|_{Z_2} = \nu(1)\lambda$. Applying the same argument to ρ_n^1 and ν we see that ν extends to a complex K_λ-character which we denote by the same letter ν. It follows that $\hat{\nu}$ and $\tilde{\mu}$ are extensions of μ to K_λ, whence $\tilde{\mu} = \hat{\nu}\gamma$ for some linear character γ of K_λ/Q_2. Now assume x is as in the lemma. Then x acts trivially on the complete inverse image of $C_{Q_2/Z_2}(x)$ in $Q_2/\operatorname{Ker}(\lambda_B)$ by Lemma 9.1, whence $\nu(x) \neq 0$ by Lemma 2.4 (applied to $Q := Q_2/\operatorname{Ker}(\lambda_B)$). Hence $|\tilde{\mu}(x)| = |\hat{\nu}(x)\gamma(x)| = |\nu(x)| \neq 0$, as stated.

If $\varepsilon = -$, then one can prove that

$$\alpha_n|_{Z_2} = \frac{(q^{n-2}-q)(q^{n-2}-1)}{2(q+1)} \cdot 1_{Z_2} + \frac{q^{2n-4}-q^{n-2}}{2}\omega_1 + q^{2n-4}\omega_2^-$$

and argue similarly. $\square$

Proposition 9.3. *Under the above assumptions, $V_2 = W_2$ as P_2-modules.*

Proof. Since $Q_2 < P_1$, $V = W$ as Q_2-modules. If V or W does not afford the orbit $\mathcal{O}_2^\varepsilon$ of Q_2-characters, then $V_2 = W_2 = 0$ and we are done. So assume that V and W both afford $\mathcal{O}_2^\varepsilon$ and fix $\lambda = \lambda_B \in \mathcal{O}_2^\varepsilon$. Clearly, $V_2|_{P_2} = \operatorname{Ind}_{K_\lambda}^{P_2}(V_\lambda)$ and similarly for W_2, hence it suffices to prove that $\varphi = \psi$, where φ, resp. ψ, is the Brauer K_λ-character afforded by the homogeneous component V_λ, resp. W_λ. Let $g \in K_\lambda$ be any ℓ'-element. Observe that there is a natural homomorphism $\pi : K_\lambda \to O(q_B) < GL(\langle e_1, e_2\rangle_{\mathbb{F}_q})$ with $\operatorname{Ker}(\pi) \simeq Q_2 : Sp_{2n-4}(q)$.

1) First we consider the case when $\pi(g)$ has even order in $O(q_B)$. Since $O(q_B)$ is a dihedral group of order $2(q-\varepsilon)$, $\pi(g)$ is an involution and hence it fixes a nonzero vector in $\langle e_1, e_2\rangle_{\mathbb{F}_q}$. Setting $X = Q_2\langle g\rangle$, we see that X is contained in a conjugate of P_1. Hence $V = W$ as X-modules, and so $V_\lambda = W_\lambda$ as X-modules by Lemma 2.3 (notice that X preserves λ). It follows that $\varphi(g) = \psi(g)$.

2) Next we consider the case when $\pi(g)$ has odd order in $O(q_B)$, i.e. $\pi(g) \in \Omega(q_B)$. Define $X := \pi^{-1}(\Omega(q_B))$. Clearly, the Q_2-module V_λ is semisimple. But $V_\lambda|_{Z_2}$ affords the Z_2-character $\dim(V_\lambda)\lambda$ and μ is the unique irreducible Q_2-character with $\mu|_{Z_2} = \mu(1)\lambda$ (see Lemma 9.2), so $V_\lambda|_{Q_2}$ is a direct sum of some copies of a simple $\mathbb{F}Q_2$-module that affords μ. Now $Q_2 \lhd K_\lambda$ and μ extends to $\tilde{\mu}$ by Lemma 9.2. It follows that $\varphi = \tilde{\mu}\alpha$ for some Brauer character α of K_λ/Q_2. Similarly, $\psi = \tilde{\mu}\beta$ for some Brauer character β of K_λ/Q_2.

Clearly, $K_\lambda = Q_2 : I_\lambda$. We can embed L_2 into $H_2 = Stab_G(\langle e_1, e_2, f_1, f_2\rangle_{\mathbb{F}_q})$, and write $X = Q_2Y$ with $Y = X \cap I_\lambda$. Observe that $Z_2Y < H_2$. Since $V = W$ as H_2-modules, $V = W$ as Z_2Y-modules. But V_λ and W_λ are the λ-homogeneous components of $V|_{Z_2}$ and $W|_{Z_2}$, and Z_2Y preserves λ. It follows by Lemma 2.3 that $V_\lambda = W_\lambda$ as Z_2Y-modules; in particular, $\tilde{\mu}(h)\alpha(h) = \varphi(h) = \psi(h) = \tilde{\mu}(h)\beta(h)$ for any ℓ'-element $h \in Y$. Recall that $h \in I_\lambda$ and h induces an element of odd order in $O(q_B)$, whence $\tilde{\mu}(h) \neq 0$ by Lemma 9.2. This implies that $\alpha(h) = \beta(h)$, i.e. $\alpha|_Y = \beta|_Y$. Therefore $\varphi(g) = \tilde{\mu}(g)\alpha(y) = \tilde{\mu}(g)\beta(y) = \psi(g)$, where $y \in gQ_2 \cap Y$. $\square$

An important role in further discussion is played by the following statement:

Proposition 9.4. *Let $G = Sp_{2n}(q)$ with $n \geq 3$ and $(n,q) \neq (3,2)$, $2 \leq j \leq n$, and let χ be a Weil character of G. Assume λ is a nontrivial linear character of Q_j/Z_j that occurs in $\chi|_{Q_j}$. Then $K_\lambda := Stab_{P_j}(\lambda)$ is contained in a conjugate of P_i for some $i \leq j-1$.*

Proof. Lemma 3.8 shows that it suffices to prove the proposition in the case $\chi \in \{\rho_n^1, \rho_n^2, \alpha_n, \beta_n\}$. We keep the notation introduced at the beginning of §3.

1) Recall that Q_j/Z_j may be identified with the additive group $M_j := M_{2n-2j,j}(\mathbb{F}_q)$. The action via conjugation of the Levi subgroup $L_j = GL_j(q) \times Sp_{2n-2j}(q)$ on M_j is given by the formula

$$gXg^{-1} = CX\,{}^tA,$$

where $g = \mathrm{diag}(A, C, {}^tA^{-1}) \in L_j$ with $A \in GL(\langle e_1, \ldots, e_j\rangle_{\mathbb{F}_q})$ and $C \in S := Sp_{2n-2j}(q)$, and $X \in M_j$. It is easy to see that any linear character of Q_j/Z_j is of the form $\lambda = \lambda_B \;:\; X \mapsto (-1)^{\mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_2}(\mathrm{Tr}({}^tBX))}$ for some $B \in M_j$. Then the action of $g = \mathrm{diag}(A, C, {}^tA^{-1}) \in L_j$ on $\mathrm{Irr}(Q_j/Z_j)$ is given by $g \circ \lambda_B = \lambda_{{}^tC^{-1}BA^{-1}}$.

From now on we assume $\lambda = \lambda_B$ satisfies the assumption of the proposition. Let $\mathcal{O}$ denote the L_j-orbit of λ. Clearly, $|\mathcal{O}| \leq \chi(1) < q^{2n}$.

2) Here we prove the statement in the case $\lambda = \lambda_B$ with $0 < r := \mathrm{rank}(B) < j$. Indeed, assume $g = \mathrm{diag}(A, C, {}^tA^{-1}) \in I_\lambda$. Then $B = {}^tC^{-1}BA^{-1}$, and so ${}^tA\,{}^tB = {}^tBC^{-1}$. Let ${}^tB = [\boldsymbol{b}_1, \ldots, \boldsymbol{b}_{2n-2j}]$, where $\boldsymbol{b}_k \in \mathbb{F}_q^j$. The formula ${}^tA\,{}^tB = {}^tBC^{-1}$ shows that ${}^tA\boldsymbol{b}_k \in \langle \boldsymbol{b}_1, \ldots, \boldsymbol{b}_{2n-2j}\rangle_{\mathbb{F}_q}$ for all k. Thus ${}^tA$ preserves a subspace of dimension r of $\mathbb{F}_q^j$, whence A preserves a subspace of dimension $j - r$ of $\mathbb{F}_q^j$. Returning to the specified basis of the natural module M of G, we see that all elements $g = \mathrm{diag}(A, C, {}^tA^{-1}) \in I_\lambda$ preserve a subspace M' (which depends only on B) of dimension $j - r$ of the totally isotropic subspace $\langle e_1, \ldots, e_j\rangle_{\mathbb{F}_q}$. Observe that $K_\lambda = Q_jI_\lambda$ and that Q_j acts trivially on $\langle e_1, \ldots, e_j\rangle_{\mathbb{F}_q}$. It follows that K_λ is contained in a conjugate of P_{j-r}.

3) Since λ is nontrivial, $B \neq 0$. The rest of the proof is to show that $\mathrm{rank}(B) \neq j$. So we will now assume that $j = \mathrm{rank}(B) \leq 2n - 2j$. We claim that there is a subspace N' of dimension j of the natural module $N = \mathbb{F}_q^{2n-2j}$ for $S = Sp_{2n-2j}(q)$ such that $|I_\lambda| = |Stab_S(N')|$. Indeed, assume $g = \mathrm{diag}(A, C, {}^tA^{-1}) \in I_\lambda$. Then $B = {}^tC^{-1}BA^{-1}$, and so ${}^tCB = BA^{-1}$. Let $B = [\boldsymbol{b}'_1, \ldots, \boldsymbol{b}'_j]$, where $\boldsymbol{b}'_k \in \mathbb{F}_q^{2n-2j}$. The formula ${}^tCB = BA^{-1}$ shows that ${}^tC\boldsymbol{b}'_k$ lies in $N' := \langle \boldsymbol{b}'_1, \ldots, \boldsymbol{b}'_j\rangle_{\mathbb{F}_q}$ for all k, i.e. ${}^tC \in Stab_S(N')$. Moreover, since $\boldsymbol{b}'_1, \ldots, \boldsymbol{b}'_j$ are linearly independent, A^{-1} (and so A) is completely determined by B and C. Thus $|I_\lambda| = |Stab_S(N')|$ as stated. In particular,

$$|\mathcal{O}| = (L_j : I_\lambda) = |GL_j(q)| \cdot (S : Stab_S(N')).$$

4) Here we handle the case $j \geq 3$. First assume that $j < 2n - 2j$. It is straightforward to check that $(S : Stab_S(N')) \geq (q^{2n-2j} - 1)/(q - 1)$. Hence $|\mathcal{O}| = |GL_j(q)| \cdot (S : Stab_S(N')) > q^{2n}$, a contradiction. Now assume that $3 \leq j = 2n - 2j$. It follows that $n \geq 6$ and $j \geq 4$. In this case $S = Stab_S(N')$, and $|\mathcal{O}| = |GL_j(q)| > q^{2n+1}$, again a contradiction.

5) We are left with the case $j = 2$. First assume that $n \geq 4$. Recall that N' is a 2-dimensional subspace of N, so $|\mathcal{O}| = |GL_2(q)| \cdot (S : Stab_S(N'))$ equals $(q^{2n-4} - 1)(q^{2n-4} - q^{2n-5})$ if N' is nondegenerate, and $(q^{2n-4} - 1)(q^{2n-5} - q)$ if N' is degenerate. In both cases $|\mathcal{O}| \geq (q^{2n-4} - 1)(q^3 - q) > q^{2n-2}$. On the other hand, it is clear that $|\mathcal{O}|$ cannot exceed the multiplicity of 1_{Z_2} in $\chi|_{Z_2}$, and, since $\chi \in \{\rho_n^1, \rho_n^2, \alpha_n, \beta_n\}$, this multiplicity is at most

$$q^{2n-3} + q^{2n-4} + (q^{n-2} + q)(q^{n-2} - 1)/2(q - 1) < q^{2n-2}.$$

Thus we get a contradiction if $n \geq 4$. Finally assume that $n = 3$, so $q \geq 4$ by our assumption. Then $|\mathcal{O}| = (q^2 - 1)(q^2 - q)$; meanwhile the multiplicity of 1_{Z_2} in $\chi|_{Z_2}$ is at most $q^3 + q^2 + q$, again a contradiction. $\square$

The main result of this section is the following theorem:

Theorem 9.5. *Let $G = Sp_{2n}(q)$ with $n \geq 3$ and $(n, q) \neq (3, 2), (4, 2)$. Assume V is an irreducible $\mathbb{F}G$-module satisfying $(\mathcal{W}_2^\varepsilon)$ for some $\varepsilon = \pm$. Then there is a formal sum W of trivial and linear-Weil modules of G if $\varepsilon = +$ and of trivial and unitary-Weil modules of G if $\varepsilon = -$, such that $V = W$ as P_j-modules for $1 \leq j \leq n-1$ and as H_d-modules for $1 \leq d \leq n/2$.*

Proof. By Theorem 8.4, there is a formal sum W of trivial and Weil modules of G such that $V = W$ as P_1-modules and as H_d-modules for $1 \leq d \leq n/2$. It remains to prove that $V = W$ as P_j-modules for $1 \leq j \leq n-1$. We proceed by induction on $j \geq 1$, with the induction base $j = 1$ already established. For the induction step assume $n > j \geq 2$ and that $V = W$ as P_i-modules for $1 \leq i \leq j-1$. We will use the decomposition (42) for V and W. Since $Q_j < P_1$, $V = W$ as Q_j-modules. Thus the irreducible Q_j-characters occurring in V and in W are exactly the same.

1) First we show that $V_0 = W_0$ as P_j-modules. Let $\lambda \in \mathrm{IBr}_\ell(Q_j/Z_j)$ be any character occurring in W_0. By Proposition 9.4, $K_\lambda := Stab_{P_j}(\lambda)$ is contained in a conjugate of some P_i with $i < j$. It follows by the induction hypothesis that $V = W$ as K_λ-modules. But $Q_j \lhd K_\lambda$ and K_λ preserves λ, so $V_\lambda = W_\lambda$ as K_λ-modules by Lemma 2.3. Let $\mathcal{O}$ be the P_j-orbit of λ and let $V_\mathcal{O} := \sum_{\mu \in \mathcal{O}} V_\mu$, $W_\mathcal{O} := \sum_{\mu \in \mathcal{O}} W_\mu$. Then $V_\mathcal{O} = \mathrm{Ind}_{K_\lambda}^{P_j}(V_\lambda) = \mathrm{Ind}_{K_\lambda}^{P_j}(W_\lambda) = W_\mathcal{O}$ as P_j-modules. This is true for any orbit $\mathcal{O}$ occurring in W_0 (and V_0), whence $V_0 = W_0$ as P_j-modules.

2) Next we show that $V_k = W_k$ as P_j-modules for $k = 1, 2$. If $k = j$, then $k = j = 2$ and we are done by Proposition 9.3. So we may assume that $k < j$. Let $\lambda = \lambda_B \in \mathrm{IBr}_\ell(Z_j)$ be any character occurring in W_k, and let $g = \mathrm{diag}(A, C, {}^tA^{-1}) \in I_\lambda$. By the definition of W_k, q_B is a quadratic form of rank k on $\mathbb{F}_q^j$; in particular $\mathrm{rad}(q_B)$ has dimension $j - k$. Since $A \in O(q_B)$, A preserves $\mathrm{rad}(q_B)$. It follows that any $g \in I_\lambda$ preserves a $(j-k)$-dimensional subspace M' of $\langle e_1, \ldots, e_j \rangle_{\mathbb{F}_q}$. It is clear that M' is totally isotropic and that Q_j acts trivially on M'. It follows that $K_\lambda := Stab_{P_j}(\lambda)$ is contained in a conjugate of P_{j-k} and so $V = W$ as K_λ-modules. But $Z_j \lhd K_\lambda$ and K_λ preserves λ, so $V_\lambda = W_\lambda$ as K_λ-modules by Lemma 2.3. Arguing as in 1) we come to the conclusion that $V_k = P_k$ as P_j-modules.

Note that 1) and 2) are also valid if $j = n$.

3) From 1) and 2) we obtain that $\bigoplus_{k=0}^2 V_k = \bigoplus_{k=0}^2 W_k$ as L_j-modules. But $L_j < H_j$ and $V = W$ as H_j-modules by the choice of W (notice that we use $j < n$ here). It now follows from (42) that $C_V(Q_j) = C_W(Q_j)$ as L_j-modules. Since Q_j acts trivially on both $C_V(Q_j)$ and $C_W(Q_j)$, they are equal as P_j-modules as well. Consequently, $V = W$ as P_j-modules. $\square$

10. Restrictions to tori T_+ and T_-

In this section we use Deligne-Lusztig theory to handle the tori T_ε of order $q^n - \varepsilon$ of G. Recall that T_ε is chosen to be a torus of $SL_2(q^n)$ which is naturally embedded in G.

10.1. **Some generalities.** We may consider G as the fixed point subgroup $\mathcal{G}^F$ for some Frobenius map F on a simple algebraic group $\mathcal{G}$ of type C_n in characteristic 2. Let $\mathcal{T}_\varepsilon$ be an F-stable maximal torus of $\mathcal{G}$ such that $T_\varepsilon = \mathcal{T}_\varepsilon^F$. We begin with the following statements, among which the first one is obvious in the case $q > 2$:

Lemma 10.1. *Let $n \geq 2$, $\mathcal{T}$ be an F-stable maximal torus of $\mathcal{G}$ and $\varepsilon = \pm$.*

(i) *Assume that $\mathcal{T}^F \leq T_\varepsilon$ but $\mathcal{T}$ is not $\mathcal{G}^F$-conjugate to $\mathcal{T}_\varepsilon$. Then $q = 2$, $\mathcal{T}$ is maximally split, and $\mathcal{T}^F = 1$.*

(ii) *Let n be an odd prime and $(n, q) \neq (3, 2)$. Assume that $|\mathcal{T}^F| = |T_\varepsilon|$. Then $\mathcal{T}$ is $\mathcal{G}^F$-conjugate to $\mathcal{T}_\varepsilon$.*

Proof. (i) In the case $q > 2$ all maximal tori of G are nondegenerate (see the proof of [C, Prop. 3.6.6]), whence $\mathcal{T}$ is the only maximal torus of $\mathcal{G}$ that contains $\mathcal{T}^F$. So if $\mathcal{T}^F \leq (\mathcal{T}')^F$ for some F-stable maximal torus $\mathcal{T}'$ of $\mathcal{G}$, then $\mathcal{T}' = \mathcal{T}$, and so we are done. We will however consider the general case.

Recall that the $\mathcal{G}^F$-classes of F-stable maximal tori of $\mathcal{G}$ are classified by F-conjugacy classes of the Weyl group $W(\mathcal{G})$ of $\mathcal{G}$; cf. [C, Prop. 3.3.3]. In our particular case, F acts trivially on $W(\mathcal{G})$, and each conjugacy class in $W(\mathcal{G})$ contains an element which is a product of positive cycles of length $a_1, \dots, a_r$ and negative cycles of length $b_1, \dots, b_s$ with $\sum_{i=1}^r a_i + \sum_{j=1}^s b_j = n$. Moreover, for the corresponding torus $\mathcal{T}$ we have $\mathcal{T}^F = \prod_{i=1}^r A_i \times \prod_{j=1}^s B_j$ with $A_i \simeq \mathbb{Z}_{q^{a_i}-1}$ and $B_j \simeq \mathbb{Z}_{q^{b_j}+1}$; see for instance [FLT, p. 16,17]. Now we assume that $\mathcal{T}^F \leq T_\varepsilon$. If $b_1 = n$, then $|\mathcal{T}^F| = q^n + 1 > |T_+|$, whence $\varepsilon = -$ and $\mathcal{T}$ is $\mathcal{G}^F$-conjugate to $\mathcal{T}_\varepsilon$, a contradiction. Assume $a_1 = n$. Then $|\mathcal{T}^F| = q^n - 1 \geq 3$ divides $|T_\varepsilon| = q^n - \varepsilon$, whence $\varepsilon = +$ and $\mathcal{T}$ is $\mathcal{G}^F$-conjugate to $\mathcal{T}_\varepsilon$, again a contradiction. We may therefore assume that $r + s \geq 2$. In this case each $g \in A_i$ has eigenvalue 1 (with multiplicity $2(n - a_i)$) on the natural module $\tilde{M}$ of $\mathcal{G}$. But the only element in T_ε with eigenvalue 1 on $\tilde{M}$ is 1. Since $A_i \leq T_\varepsilon$, it follows that $A_i = 1$, i.e. $q = 2$ and $a_i = 1$. The same argument shows that there is actually no B_j in $\mathcal{T}^F$. We conclude that $q = 2$, $\mathcal{T}^F$ is maximally split, and $\mathcal{T}^F = 1$.

(ii) First assume that $n = 3$. Then there are 10 $\mathcal{G}^F$-classes of F-stable maximal tori $\mathcal{T}$ in $\mathcal{G}$, with $|\mathcal{T}^F| = (q-1)^3$, $(q-1)^2(q+1)$ (2 classes), $(q+1)^2(q-1)$ (2 classes), $(q-1)(q^2+1)$, $(q+1)(q^2+1)$, $(q+1)^3$, $q^3 - 1$, and $q^3 + 1$; cf. [Lu, p. 87]. Since $q \geq 4$, it is easy to check that these 8 orders are pairwise distinct, and so we are done. Next assume that $n \geq 5$, $|\mathcal{T}^F| = |T_\varepsilon|$, and we keep the notation in the proof of (i) for $\mathcal{T}^F$. Then $q^n - \varepsilon = \prod_{i=1}^r (q^{a_i} - 1) \cdot \prod_{j=1}^s (q^{b_j} + 1)$. If $a_1 = n$, then $\varepsilon = +$ and $\mathcal{T}$ is $\mathcal{G}^F$-conjugate to $\mathcal{T}_\varepsilon$. Similarly, if $b_1 = n$, then $\varepsilon = -$ and $\mathcal{T}$ is $\mathcal{G}^F$-conjugate to $\mathcal{T}_\varepsilon$. So we may assume $r + s \geq 2$; in particular, $a_i, b_j < n$ for all i, j. Since $n \geq 5$ is a prime, by [Zs] there is a prime divisor r of $q^n - 1$ which does not divide $\prod_{i=1}^{n-1}(q^i - 1)$ if $\varepsilon = +$, and a prime divisor r of $q^{2n} - 1$ which does not divide $\prod_{i=1}^{2n-1}(q^i - 1)$ if $\varepsilon = -$. Clearly, r divides $|T_\varepsilon|$ but not $|\mathcal{T}^F|$, a contradiction. $\square$

Recall that to each F-stable maximal torus $\mathcal{T}$ of $\mathcal{G}$ and a character $\theta \in \mathrm{Irr}(\mathcal{T}^F)$ one can define the Deligne-Lusztig (virtual) character $R_{\mathcal{T},\theta}$; cf. [C, DM]. The characters $R_{\mathcal{T},\theta}$ are parametrized by the $\mathcal{G}^F$-conjugacy classes of pairs $(\mathcal{T}, \theta)$. Let $\mathcal{G}^*$ be a simple algebraic group with a Frobenius map F^* such that $(\mathcal{G}^*, F^*)$ is dual to $(\mathcal{G}, F)$. Then the $\mathcal{G}^F$-conjugacy classes of $(\mathcal{T}, \theta)$ are in a bijective correspondence Π with the $\mathcal{G}^{*F^*}$-conjugacy classes of pairs $(\mathcal{T}^*, s)$, where $s \in \mathcal{G}^{*F^*}$ is semisimple and $\mathcal{T}^*$ is a F^*-stable maximal torus containing s [DM, Prop. 13.13]. This correspondence Π is explicitly described in [Lu, p. 23, 24]. The set $\mathrm{Irr}(\mathcal{G}^F)$ is partitioned into Lusztig series $\mathcal{E}(\mathcal{G}^F, (s))$ (cf. [DM, p. 107]) which run over all geometric conjugacy classes (s) of semisimple elements $s \in \mathcal{G}^{*F^*}$. On the other hand, if $s \in \mathcal{G}^{*F^*}$ is semisimple, then the rational series $\mathcal{E}(\mathcal{G}^F, (s)_{\mathcal{G}^{*F^*}})$ consists of all irreducible constituents of those $R_{\mathcal{T},\theta}$ such that $(\mathcal{T},\theta)$ correspond to $(\mathcal{T}^*, s')$ with s' being $\mathcal{G}^{*F^*}$-conjugate to s. (Informally speaking, we say that such $R_{\mathcal{T},\theta}$ belong to $\mathcal{E}(\mathcal{G}^F,(s)_{\mathcal{G}^{*F^*}})$.) In our particular situation, since $Z(\mathcal{G})$ is connected, the Lusztig series $\mathcal{E}(\mathcal{G}^F,(s))$ and the rational series $\mathcal{E}(\mathcal{G}^F,(s)_{\mathcal{G}^{*F^*}})$ are the same (cf. [DM, p. 136]), hence we will use the notation $\mathcal{E}(\mathcal{G}^F,(s))$ to denote the corresponding rational series.

In general, the Deligne-Lusztig characters $R_{\mathcal{T},\theta}$ span only a subspace of the space of all class functions on G, and any class function on G that is orthogonal to all Deligne-Lusztig characters (with respect to the usual scalar product $(\cdot,\cdot)_G$) is called an orthogonal function (Senkrechtfunktion in [Lu]).

Lemma 10.2. (i) *Let f be any orthogonal function. Then $f(t) = 0$ for any semisimple element $t \in G$.*

(ii) *Assume $Z(\mathcal{G})$ is connected and $\chi \in \mathrm{Irr}(\mathcal{G}^F) \cap \mathcal{E}(\mathcal{G}^F,(s))$. Then χ is a linear combination of those $R_{\mathcal{T},\theta}$ belonging to $\mathcal{E}(\mathcal{G}^F,(s))$ and some orthogonal functions.*

Proof. (i) Let $t \in G$ be semisimple and let ψ be the class function which takes value 1 on t^G and value 0 otherwise. By [C, Prop. 7.5.5], ψ is a uniform function, hence ψ is orthogonal to f. Thus

$$0 = (f,\psi)_G = \frac{1}{|G|}\sum_{g\in G} f(g)\overline{\psi(g)} = \frac{1}{|G|}\sum_{g\in t^G} f(g) = \frac{|t^G|\cdot f(t)}{|G|} = \frac{f(t)}{|C_G(t)|},$$

i.e. $f(t) = 0$.

(ii) Let $I := \mathrm{Irr}(\mathcal{G}^F)\cap\mathcal{E}(\mathcal{G}^F,(s))$, $I' := \mathrm{Irr}(\mathcal{G}^F)\setminus I$, $J := \{R_{\mathcal{T},\theta} \mid R_{\mathcal{T},\theta} \in \mathcal{E}(\mathcal{G}^F,(s))\}$, and $J' := \{R_{\mathcal{T},\theta} \mid R_{\mathcal{T},\theta} \notin \mathcal{E}(\mathcal{G}^F,(s))\}$. Clearly, $I \perp I'$, $J \subseteq \langle I\rangle_{\mathbb{C}}$, and $J' \subseteq \langle I'\rangle_{\mathbb{C}}$ (as Lusztig series form a partition of $\mathrm{Irr}(\mathcal{G}^F)$). Write $\langle I\rangle_{\mathbb{C}}$ as the orthogonal sum $\langle J\rangle_{\mathbb{C}} \oplus S$. Then any function $f \in S$ is orthogonal to J and also to J', so f is an orthogonal function. Now any $\chi \in I$ can be written as $\alpha+\beta$, where $\alpha \in \langle J\rangle_{\mathbb{C}}$ and $\beta \in S$, and so we are done. □

Remark 10.3. Assume that $(\mathcal{G},F)$ and $(\mathcal{G}^*,F^*)$ are dual to each other, with corresponding dual rational maximal tori $\mathcal{T}$ and $\mathcal{T}^*$. Then $\mathcal{T}^{*F^*}$ is isomorphic to $\mathrm{Irr}(\mathcal{T}^F)$ (considered under multiplication) (cf. [DM, Prop. 13.11]) and the above correspondence Π specifies an isomorphism between them.

10.2. **The case n is an odd prime.** Throughout this subsection we assume that n is an odd prime and $(n,q) \neq (3,2)$. Let G_ℓ denote the set of G-conjugacy classes of ℓ'-elements. The torus T_+ has a unique subgroup $T_+^1 \simeq \mathbb{Z}_{q-1}$, and similarly the torus T_- has a a unique subgroup $T_-^1 \simeq \mathbb{Z}_{q+1}$. Let $\mathcal{X}_\ell$ denote the set of G-conjugacy classes of ℓ'-elements that intersect either $T_+ \setminus T_+^1$ or $T_- \setminus T_-^1$. Since n is assumed to be an odd prime, one can check that the elements in $T_+\setminus T_+^1$ split into $\frac{1}{2n}(q^n-q)$ G-classes, with representatives $h_{28}(i)$, $1 \leq i \leq (q^n-q)/2n$ (where we keep the notation consistent with the one used in [Lu] for $n=3$), and the centralizer of any of them in G equals T_+. Similarly, the elements in $T_-\setminus T_-^1$ split into $\frac{1}{2n}(q^n-q)$ G-classes, with representatives $h_{31}(i)$, $1 \leq i \leq (q^n-q)/2n$, and the centralizer of any of them in G equals T_-. The elements in $T_+^1\setminus\{1\}$ split into $\frac{q-2}{2}$ G-classes, with representatives $h_5(i)$, $1 \leq i \leq (q-2)/2$, and the elements in $T_-^1\setminus\{1\}$ split into $\frac{q}{2}$ G-classes, with representatives $h_6(i)$, $1 \leq i \leq q/2$.

In our case $\mathcal{G}$ is of type C_n and q is even, so $\mathcal{G}^F \simeq \mathcal{G}^{*F^*}$. Let $g_8(i)$, resp. $g_9(i)$, $g_{31}(i)$, $g_{34}(i)$, be representatives of the $\mathcal{G}^{*F^*}$-classes corresponding to the G-classes of $h_5(i)$, $h_6(i)$, $h_{28}(i)$, $h_{31}(i)$, respectively (again we keep the notation consistent with the one used in [Lu] for $n = 3$). In particular, $C_{\mathcal{G}^{*F^*}}(g_k(i))$ is isomorphic to $GL_n(q)$, resp. $GU_n(q)$, T_+, or T_-, if $k = 8$, 9, 31, or 34, resp.

For any semisimple ℓ'-element $s \in \mathcal{G}^{*F^*}$, let

$$\mathcal{E}_\ell(\mathcal{G}^F, (s)) = \bigcup_{t \in C_{\mathcal{G}^{*F^*}}(s),\ t \text{ any } \ell\text{-element}} \mathcal{E}(\mathcal{G}^F, st).$$

A fundamental result of Broué and Michel [BM] asserts that $\mathcal{E}_\ell(\mathcal{G}^F, (s))$ is a union of ℓ-blocks. Abusing notation, we will also let $\mathcal{E}_\ell(\mathcal{G}^F, (s))$ denote the set of all irreducible ℓ-Brauer characters that belong to this union of ℓ-blocks.

Let S_ℓ be the set of ℓ'-elements $1 \neq s \in \mathcal{G}^{*F^*}$ that are $\mathcal{G}^{*F^*}$-conjugate to any of the above elements $g_8(i)$, $g_9(i)$, $g_{31}(i)$, $g_{34}(i)$, and let

$$\mathcal{S}_\ell = \left(\bigcup_{s \in S_\ell} \mathcal{E}_\ell(\mathcal{G}^F, (s)) \right) \cap \mathrm{IBr}_\ell(\mathcal{G}^F).$$

Lemma 10.4. *Let $G = Sp_{2n}(q)$, n an odd prime, $(n, q) \neq (3, 2)$, and keep the above notation. The restriction of functions from $\mathcal{S}_\ell$ to $\mathcal{X}_\ell$ spans the space F of class functions on $\mathcal{X}_\ell$.*

Proof. Abusing notation, in this proof we denote by $\hat{\chi}$ the restriction of any class function χ on G or on G_ℓ to $\mathcal{X}_\ell$. Clearly, F is spanned by $\{\hat{\psi} \mid \psi \in \mathrm{IBr}_\ell(G)\}$. By a result of Geck and Hiss [GH], $\{\hat{\chi} \mid \chi \in \mathcal{E}(\mathcal{G}^F, (s)) \cap \mathrm{Irr}(G)\}$ form a basic set for the Brauer characters in $\mathcal{E}_\ell(\mathcal{G}^F, (s))$. Hence it suffices to show that for any semisimple ℓ'-element $s \in \mathcal{G}^{*F^*}$ and any $\chi \in \mathcal{E}(\mathcal{G}^F, (s))$, $\hat{\chi}$ belongs to $F_0 := \langle \hat{\psi} \mid \psi \in \mathcal{S}_\ell \rangle_{\mathbb{C}}$.

1) All classes in $\mathcal{X}_\ell$ are semisimple, whence all orthogonal functions vanish at them by Lemma 10.2(i). Furthermore, if $\mathcal{T}$ is not $\mathcal{G}^F$-conjugate to $\mathcal{T}_\varepsilon$ for any ε, then $R_{\mathcal{T},\theta}(g) = 0$ for any $g \in \mathcal{X}_\ell$. For, assume the contrary: $R_{\mathcal{T},\theta}(g) \neq 0$ for some $g \in \mathcal{X}_\ell$. Since $C_{\mathcal{G}}(g)$ is connected, [C, Theorem 7.2.8] yields

$$R_{\mathcal{T},\theta}(g) = \frac{1}{|C_G(g)|} \sum_{x \in G,\ x^{-1}gx \in \mathcal{T}^F} \theta(x^{-1}gx) Q^{C_{\mathcal{G}}(g)}_{x\mathcal{T}x^{-1}}(1). \tag{43}$$

Since $R_{\mathcal{T},\theta}(g) \neq 0$, it follows that there is an $x \in G$ such that $x^{-1}gx \in \mathcal{T}^F$. Without loss we may replace g by $x^{-1}gx$ and assume $g \in \mathcal{T}^F$. In this case $\mathcal{T}^F \leq C_G(g) = T_\varepsilon$ for some $\varepsilon = \pm$. By Lemma 10.1(i), $\mathcal{T}^F = 1$, contrary to the fact that $\mathcal{T}^F \ni g \neq 1$.

2) Here we consider the case $1 \neq s \notin S_\ell$. We claim that for any $R_{\mathcal{T},\theta}$ in $\mathcal{E}(\mathcal{G}^F, (s))$, $\mathcal{T}$ is not $\mathcal{G}^F$-conjugate to any $\mathcal{T}_\varepsilon$. Indeed, assume to the contrary that $\mathcal{T}$ is $\mathcal{G}^F$-conjugate to $\mathcal{T}_\varepsilon$ for some $\varepsilon = \pm$. The above correspondence Π sends $(\mathcal{T}, \theta)$ to $(\mathcal{T}^*, s)$ and gives an isomorphism $\mathrm{Irr}(\mathcal{T}^F) \simeq \mathcal{T}^{*F^*}$; cf. Remark 10.3. Now $|\mathcal{T}^{*F^*}| = |\mathrm{Irr}(\mathcal{T}^F)| = |\mathcal{T}^F| = |T_\varepsilon|$. By Lemma 10.1(ii) (applied to $\mathcal{G}^*$), there is a unique $\mathcal{G}^{*F^*}$-class of such $\mathcal{T}^*$, and this is the class corresponding to $\mathcal{T}_\varepsilon$. But $s \in \mathcal{T}^{*F^*}$, so in this case s is $\mathcal{G}^{*F^*}$-conjugate either to 1 or to an element of S_ℓ, contrary to the assumption we made on s.

Now let $\chi \in \mathcal{E}(\mathcal{G}^F, (s))$. By Lemma 10.2(ii), χ is a linear combination of some orthogonal functions and some $R_{\mathcal{T},\theta}$ belonging to $\mathcal{E}(\mathcal{G}^F, (s))$. According to 1), the orthogonal functions all vanish on $\mathcal{X}_\ell$, and the $R_{\mathcal{T},\theta}$'s also vanish on $\mathcal{X}_\ell$ since $\mathcal{T}$ is not $\mathcal{G}^F$-conjugate to any $\mathcal{T}_\varepsilon$. Hence $\hat{\chi} = 0$.

3) If $s \in S_\ell$, then clearly $\hat{\chi} \in F_0$. Finally, we consider the case $s = 1$. By Lemma 10.2(ii), χ is a linear combination of some orthogonal functions and some

$R_{\mathcal{T},1}$. Again, orthogonal functions vanish on $\mathcal{X}_\ell$, and $R_{\mathcal{T},1}$ vanishes on $\mathcal{X}_\ell$ if $\mathcal{T}$ is not $\mathcal{G}^F$-conjugate to any $\mathcal{T}_\varepsilon$. So it suffices to show that $\hat{R}_{\mathcal{T},1}$ belongs to F_0 when $\mathcal{T}$ is $\mathcal{G}^F$-conjugate to $\mathcal{T}_\varepsilon$.

Assume for instance that $\mathcal{T}$ is $\mathcal{G}^F$-conjugate to $\mathcal{T}_+$. We may assume that $\mathcal{T}^F = T_+$. Let t be a generator of $T_+ \simeq \mathbb{Z}_{q^n-1}$, $\tilde{t}$ be a primitive $(q^n-1)^{\text{th}}$-root of unity in $\mathbb{C}$, and write $q^n - 1 = \ell^a r$ with $(r,\ell) = 1$. For $0 \leq i \leq r-1$, let θ_i be the T_+-character sending t to $\tilde{t}^{\ell^a i}$. Let $g \in T_- \cap \mathcal{X}_\ell$. Since $(|T_-|, |T_+|) = 1$, no $\mathcal{G}^F$-conjugate of g can belong to T_+. Then (43) implies that $R_{\mathcal{T},1}$ and $\sum_{i=0}^{r-1} R_{\mathcal{T},\theta_i}$ vanish at g.

First we assume $r = 1$, i.e. $q^n - 1 = \ell^a$. Then every element $h \in \mathcal{X}_\ell$ is conjugate to an element $g \in T_-$, hence $R_{\mathcal{T},1}(h) = 0$ due to the above observation. Thus $\hat{R}_{\mathcal{T},1} = 0$.

Next we assume that $r > 1$. We have shown above that $\sum_{i=0}^{r-1} R_{\mathcal{T},\theta_i}$ vanishes at any element $g \in T_- \cap \mathcal{X}_\ell$. Let $h \in T_+ \cap \mathcal{X}_\ell$. Then $h = t^{\ell^a k}$ with $1 \leq k \leq r-1$, and so

$$\sum_{i=0}^{r-1} \theta_i(h) = \sum_{i=0}^{r-1} \tilde{t}^{\ell^{2a} ki} = \frac{\tilde{t}^{\ell^{2a} kr} - 1}{\tilde{t}^{\ell^{2a} k} - 1} = 0.$$

Applying (43) to h we get

$$\sum_{i=0}^{r-1} R_{\mathcal{T},\theta_i}(h) = \frac{1}{|C_G(h)|} \sum_{x \in G,\ x^{-1}hx \in T_+} \left(\sum_{i=0}^{r-1} \theta_i(x^{-1}hx) \right) Q^{C_{\mathcal{G}}(h)}_{x\mathcal{T}x^{-1}}(1) = 0.$$

Thus $\sum_{i=0}^{r-1} R_{\mathcal{T},\theta_i}$ vanishes at any element $h \in T_+ \cap \mathcal{X}_\ell$. Consequently,

$$\hat{R}_{\mathcal{T},1} = -\sum_{i=1}^{r-1} \hat{R}_{\mathcal{T},\theta_i}. \tag{44}$$

Observe that for any i, $1 \leq i \leq r-1$, θ_i is a nontrivial character of $\mathcal{T}^F$ of ℓ'-order (indeed, $\theta_i^r = 1_{\mathcal{T}^F}$). So the correspondence Π sends $(\mathcal{T}, \theta_i)$ to $(\mathcal{T}^*, s')$, where $s' \in \mathcal{T}^{*F^*}$ is a nontrivial ℓ'-element. As in 2), the equality $q^n - 1 = |\mathcal{T}^F| = |\mathcal{T}^{*F^*}|$ implies that $\mathcal{T}^*$ is $\mathcal{G}^{*F^*}$-conjugate to the torus corresponding to $\mathcal{T}_+$. Since s' is a nontrivial ℓ'-element in $\mathcal{T}^{*F^*}$, s' is $\mathcal{G}^{*F^*}$-conjugate to $g_8(l)$ or $g_{31}(l)$ (for some l). In other words, $s' \in S_\ell$. Now $R_{\mathcal{T},\theta_i}$ is certainly a linear combination of some $\chi \in \mathcal{E}(\mathcal{G}^F, (s')) \cap \mathrm{Irr}(G)$, whence $\hat{R}_{\mathcal{T},\theta_i}$ is a linear combination of some $\hat{\psi}$ with $\psi \in \mathcal{E}_\ell(\mathcal{G}^F, (s')) \cap \mathrm{IBr}_\ell(G)$. It follows that $\hat{R}_{\mathcal{T},\theta_i} \in F_0$ for any i, $1 \leq i \leq r-1$. But in this case (44) implies that $\hat{R}_{\mathcal{T},1} \in F_0$ as well.

The case when $\mathcal{T}$ is $\mathcal{G}^F$-conjugate to $\mathcal{T}_-$ can be dealt with in the same way. □

A crucial ingredient of our argument is

Proposition 10.5. *Let $G = Sp_{2n}(q)$, n an odd prime, $(n,q) \neq (3,2)$, and keep the above notation. Let φ be a class functions on G_ℓ. Assume that $\varphi \in \langle \mathrm{IBr}_\ell(G) \setminus \mathcal{S}_\ell \rangle_{\mathbb{C}}$ and that $\varphi = 0$ on $G_\ell \setminus \mathcal{X}_\ell$. Then $\varphi = 0$.*

Proof. Let $g_1, \ldots, g_m$ be representatives of all G-classes in $\mathcal{X}_\ell$. By Lemma 10.4, there are $\psi_1, \ldots, \psi_m \in \langle \mathcal{S}_\ell \rangle_{\mathbb{C}}$ such that $\psi_i(g_j) = \delta_{ij}$. The aforementioned result of Broué and Michel [BM] implies that $\mathrm{IBr}_\ell(G) \setminus \mathcal{S}_\ell$ and $\mathcal{S}_\ell$ are disjoint unions of ℓ-blocks. Hence the orthogonality relations for Brauer characters yield for any i

that

$$
\begin{aligned}
0 = (\varphi, \psi_i)'_G &:= \frac{1}{|G|} \sum_{x \in G_\ell} \varphi(x)\overline{\psi_i(x)} \\
&= \frac{1}{|G|} \sum_{x \in \mathcal{X}_\ell} \varphi(x)\overline{\psi_i(x)} \\
&= \frac{1}{|G|} \sum_{x \in g_i^G} \varphi(x) = \frac{|g_i^G|}{|G|}\,\varphi(g_i),
\end{aligned}
$$

i.e. $\varphi(g_i) = 0$ for any i. Thus $\varphi = 0$ on $\mathcal{X}_\ell$, whence $\varphi = 0$. □

Theorem 10.6. *Let $G = Sp_{2n}(q)$, n an odd prime, $(n, q) \neq (3, 2)$. Assume that V and W satisfy the assumptions (and therefore the conclusions) of Theorem 9.5. Then $V = W$ and V is either the trivial module or a Weil module.*

Proof. 1) Again we view $G = \mathcal{G}^F$ for some Frobenius map F. Then we can find a rational parabolic subgroup $\mathcal{P}$ and its rational Levi subgroup $\mathcal{L}$ such that $P_1 = \mathcal{P}^F$ and $L_1 = \mathcal{L}^F$. In this case, the Lusztig functor $R^{\mathcal{G}}_{\mathcal{L}}$ is just the Harish-Chandra induction $R^G_{L_1}$, and the Lusztig restriction ${}^*R^{\mathcal{G}}_{\mathcal{L}}$ is the Harish-Chandra restriction ${}^*R^G_{L_1}$; cf. [DM, p. 48]. Since $Z(\mathcal{G})$ is connected, the Lusztig functor $R^{\mathcal{G}}_{\mathcal{L}}$ respects Lusztig series, i.e. if $s \in \mathcal{L}^{*F^*}$ and $\chi \in \mathcal{E}(\mathcal{L}^F, (s))$, then $R^{\mathcal{G}}_{\mathcal{L}}(\chi)$ is a $\mathbb{Z}$-combination of characters from $\mathcal{E}(\mathcal{G}^F, (s))$; cf. [Lu, p. 70].

2) First we consider the case when V satisfies $(\mathcal{W}_2^-)$ and assume $C_V(Q_1) = 0$. Then (30) and Corollary 7.5 imply that $a = b = c_i = 0$ and so $d = 0$ (in the notation of the proof of Proposition 8.1), whence $W = 0$ (see the definition of W right after (27)), and so $\dim(V) = \dim(W) = 0$, a contradiction. Similarly, assume that V satisfies $(\mathcal{W}_2^+)$ and $C_V(Q_1) = 0$. Then $C_W(Q_1) = C_V(Q_1) = 0$, whence (37) and Corollary 7.10 imply that $a = b = c_i = 0$ and so $d = 0$ (in the notation of the proof of Proposition 8.3). It follows that $W = 0$ (see the definition of W right after (32)), and so $\dim(V) = \dim(W) = 0$, again a contradiction.

We have shown that ${}^*R^G_{L_1}(V) = C_V(Q_1) \neq 0$. Since V is an $\mathbb{F}G$-module, we can find an L_1-simple submodule U of ${}^*R^G_{L_1}(V)$. By Frobenius' reciprocity, V is a quotient of $R^G_{L_1}(U)$. The discussion in 1) shows that $R^G_{L_1}$ respects Lusztig series.

3) Assume V satisfies $(\mathcal{W}_2^-)$. By (30) and Corollary 7.5, U can be chosen to have Brauer character $\hat{\rho} - e \cdot 1_{L_1}$, where $\rho = 1_{L_1}$, $\widehat{\alpha}_{n-1} \otimes 1_{T_1}$, $\widehat{\beta}_{n-1} \otimes 1_{T_1}$, or $\hat{\zeta}^i_{n-1} \otimes 1_{T_1}$, and $e \in \mathbb{Z}$. In the first three cases, ρ is a unipotent character of L_1; define $s \in \mathcal{L}^{*F^*}$ to be 1. In the last case, item 2) in the proof of Corollary 7.5 shows that $\rho \in \mathcal{E}(\mathcal{L}^F, (s))$, where s corresponds to some element of the form $\mathrm{diag}(\xi^j, \xi^{-j}, I_{2n-4})$ of L'_1. Clearly, in all cases the ℓ'-part of s does not belong to S_ℓ. Hence, V belongs to $\mathrm{IBr}_\ell(G) \setminus \mathcal{S}_\ell$. Also, recall that W is defined to be the virtual $\mathbb{F}G$-module, with character $a\widehat{\alpha}_n + b\widehat{\beta}_n + \sum_{i=1}^{(r-1)/2} c_i \hat{\zeta}^i_n + d \cdot 1_G$. Clearly, $\widehat{\alpha}_n$, $\widehat{\beta}_n$, and 1_G are unipotent, and $\hat{\zeta}^i_n$ is in $\mathcal{E}_\ell(\mathcal{G}^F, (s'))$ with s' corresponds to some element of the form $\mathrm{diag}(\xi^{j'}, \xi^{-j'}, I_{2n-2})$ of G, whence all irreducible constituents of them belong to $\mathrm{IBr}_\ell(G) \setminus \mathcal{S}_\ell$.

4) Assume V satisfies $(\mathcal{W}_2^+)$. By (37) and Corollary 7.10, U can be chosen to have Brauer character $\hat{\rho} - e \cdot 1_{L_1}$, where $\rho = 1_{L'_1} \otimes \delta_i$, 1_{L_1}, $\hat{\rho}^1_{n-1} \otimes 1_{T_1}$, $\hat{\rho}^2_{n-1} \otimes 1_{T_1}$, $\hat{\tau}^i_{n-1} \otimes 1_{T_1}$, and $e \in \mathbb{Z}$. In the first case, V is a quotient of $R^G_{L_1}(1_{L'_1} \otimes \delta_i) = \tau^i_n$,

so V belongs to $\mathrm{IBr}_\ell(G) \setminus \mathcal{S}_\ell$. In the next three cases, ρ is a unipotent character of L_1; define $s \in \mathcal{L}^{*F^*}$ to be 1. In the last case, item 2) in the proof of Corollary 7.10 shows that $\rho \in \mathcal{E}(\mathcal{L}^F, (s))$, where s corresponds to some element of the form $\mathrm{diag}(\delta^j, \delta^{-j}, I_{2n-4})$ of L'_1. Clearly, in the last four cases the ℓ'-part of s does not belong to S_ℓ. Hence, V belongs to $\mathrm{IBr}_\ell(G) \setminus \mathcal{S}_\ell$. Also, recall that W is defined to be the virtual $\mathbb{F}G$-module, with character

$$a\hat{\rho}_n^1 + b\hat{\rho}_n^2 + \sum_{i=1}^{(r-1)/2} c_i \hat{\tau}_n^i + (d - a - b - 2\sum_{i=1}^{(r-1)/2} c_i) \cdot 1_G.$$

Clearly, $\hat{\rho}_n^1$, $\hat{\rho}_n^2$, and 1_G are unipotent, and $\hat{\tau}_n^i$ is in $\mathcal{E}_\ell(\mathcal{G}^F, (s'))$ with s' corresponding to some element of the form $\mathrm{diag}(\delta^{j'}, \delta^{-j'}, I_{2n-2})$ of G, whence all irreducible constituents of them belong to $\mathrm{IBr}_\ell(G) \setminus \mathcal{S}_\ell$.

5) We have shown in 3) and 4) that $\varphi, \psi \in \langle \mathrm{IBr}_\ell(G) \setminus \mathcal{S}_\ell \rangle_\mathbb{C}$ if φ is the Brauer character of V and ψ is the Brauer character of W. By the conclusion of Theorem 9.5, $\varphi = \psi$ on any P_j with $1 \leq j \leq n-1$ and on any H_d with $1 \leq d \leq n/2$. Observe that any element $g \in T_+^1 \cup T_-^1$ lies in a conjugate of H_1. From Lemma 2.2(iii), it now follows that $\varphi = \psi$ on $G_\ell \setminus \mathcal{X}_\ell$. Applying Proposition 10.5 to $\varphi - \psi$, we conclude that $\varphi = \psi$, i.e. $V = W$. Since V is irreducible, the last equality implies that V is either the trivial module or a Weil module. □

10.3. **Proof of the main theorems.** We can now complete the proof of the main results of the paper.

Proof of Theorem 1.2. The cases $n = 2$ and $(n, q) = (4, 2)$ follow from Proposition 4.1 and Lemma 4.3. Furthermore, the case n is an odd prime follows from Theorems 9.5 and 10.6. We may now assume that n is a composite. Choose $r = 2$ if n is even and r to be an odd prime divisor of n if n is odd; in particular, $n = rm$ for some $m > 1$.

By Theorem 9.5, there is a formal sum W of trivial and Weil modules of G such that $V = W$ on subgroups P_j and H_d for $1 \leq j \leq n-1$ and $1 \leq d \leq n/2$. Embed $S := Sp_{2r}(q^m)$ naturally in G, and define $\kappa = \varepsilon^m$. By Lemma 3.7, any composition factor of $W|_S$ is either a trivial module, or a linear-Weil module (of S) when $\kappa = +$ and a unitary-Weil module when $\kappa = -$. By Corollary 4.2, $W|_S$ has property $(\mathcal{W}_2^\kappa)$. Consider the Z_r-subgroup $Z_r(S)$ of S. Then all $Z_r(S)$-characters occurring in $W|_{Z_r(S)}$ belong to $\mathcal{O}_1 \cup \mathcal{O}_2^\kappa$. Since $Z_r(S) < P_1$ and $V = W$ as P_1-modules, the same holds for $V|_{Z_r(S)}$, whence the S-module V also satisfies $(\mathcal{W}_2^\kappa)$. Observe that $q^m > 2$. Hence Proposition 4.1, resp. Theorems 9.5 and Theorem 10.6, applies to S when $r = 2$, resp. when $r > 2$, and yields that any composition factor of $V|_S$ is either a trivial module, or a linear-Weil module (of S) when $\kappa = +$ and a unitary-Weil module when $\kappa = -$. Now we consider the P_1-subgroup $P_1(S)$ of S. Then $P_1(S)$ is contained in a conjugate of P_m, and $1 \leq m = n/r < n$, whence $V = W$ as $P_1(S)$-modules. Thus the S-Brauer character φ of $V - W$ satisfies the assumptions of Lemma 7.11(i) when $\kappa = +$, and of Lemma 7.11(ii) when $\kappa = -$. By Lemma 7.11, $\varphi = 0$, i.e. $V = W$ as S-modules.

Finally, by Lemma 2.2, any element $g \in G$ is conjugate to an element of some P_j with $1 \leq j \leq n-1$, some H_d with $1 \leq d \leq n/2$, or S. We conclude that the Brauer characters of V and of W take the same value at any ℓ'-element $g \in G$, i.e.

$V = W$. Since V is irreducible, the last equality implies that V is either a trivial module or a Weil module. □

Proof of Theorem 1.1. The cases $(n, q) = (2, 2), (3, 2), (4, 2)$ can be checked directly using [JLPW], so we assume that $q \geq 4$ if $n \leq 4$.

First we consider the case $n \geq 5$. If $(n, q) = (5, 2)$, then $\mathfrak{d}(n, q) = 3808$. If $(n, q) \neq (5, 2)$, then $\mathfrak{d}(n, q) = b^-(n-1, q) \cdot q^{n-1}(q^{n-1} - 1)(q-1)/2$ (in the notation of Theorem 5.12). By Theorem 5.12 and Remark 5.13, V satisfies $(\mathcal{W}_1)$, whence V satisfies $(\mathcal{W}_2^\varepsilon)$ for some $\varepsilon = \pm$ by Theorem 5.11. But in this case V is either the trivial module or a Weil module by Theorem 1.2.

Next we assume that $n \leq 4$. If V satisfies $(\mathcal{W}_2^\varepsilon)$, then we are done by Theorem 1.2. Assume neither $(\mathcal{W}_2^+)$ nor $(\mathcal{W}_2^-)$ holds for V. By Lemma 3.4(i), both $(\mathcal{W}_2^+)$ and $(\mathcal{W}_2^-)$ fail for $V|_{Z_n}$. In other words, $V|_{Z_n}$ affords both $\mathcal{O}_2^+$ and $\mathcal{O}_2^-$. It follows that $\dim(V) \geq |\mathcal{O}_2^+| + |\mathcal{O}_2^-| = \mathfrak{d}(n, q)$, a contradiction. □

Remark 10.7. C. Bonnafé and R. Rouquier (private communication) have recently announced a result establishing a Morita equivalence between nonisolated blocks in $\mathcal{E}_\ell(\mathcal{G}^F, (s))$ and $\mathcal{E}_\ell(\mathcal{L}^F, (s))$ when $s \in \mathcal{L}^{*F^*}$. Suppose that V has property $(\mathcal{W}_2^\varepsilon)$. Then Propositions 8.1 and 8.3 and arguments as in the proof of Theorem 10.6 show that either V is in a unipotent block or it is in a nonisolated block. If V is in a nonisolated block, then the announced result of Bonnafé and Rouquier would imply that either V is a Weil module, or $\dim(V) > \mathfrak{d}(n, q)$ (but then one still has to show that the property $(\mathcal{W}_2^\varepsilon)$ implies $\dim(V) < \mathfrak{d}(n, q)$). More importantly, unipotent blocks are isolated, and, as it happens most of the time, dealing with unipotent blocks is the most difficult step in solving various problems in the cross characteristic representation theory of finite groups of Lie type.

References

[Atlas] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, '*An ATLAS of Finite Groups*', Clarendon Press, Oxford, 1985. MR **88g:**20025

[BM] M. Broué and J. Michel, Blocs et séries de Lusztig dans un groupe réductif fini, *J. reine angew. Math.* **395** (1989), 56 - 67. MR **90b:**20037

[BrK] J. Brundan and A. S. Kleshchev, Lower bounds for the degrees of irreducible Brauer characters of finite general linear groups, *J. Algebra* **223** (2000), 615 - 629. MR **2001f:**20014

[C] R. Carter, '*Finite Groups of Lie type: Conjugacy Classes and Complex Characters*', Wiley, Chichester, 1985. MR **87d:**20060

[DM] F. Digne and J. Michel, '*Representations of Finite Groups of Lie Type*', London Mathematical Society Student Texts 21, Cambridge University Press, 1991. MR **92g:**20063

[DT] N. Dummigan and Pham Huu Tiep, Lower bounds for the minima of certain symplectic and unitary group lattices, *Amer. J. Math.* **121** (1999), 889 - 918. MR **2001a:**11112

[E] H. Enomoto, The characters of the finite symplectic groups $Sp(4, q)$, $q = 2^f$, *Osaka J. Math.* **9** (1972), 75 - 94. MR **46:**1893

[FLT] P. Fleischmann, W. Lempken, and Pham Huu Tiep, The p-intersection subgroups in quasi-simple and almost simple finite groups, *J. Algebra* **207** (1998), 1 - 42. MR **99j:**20021

[GH] M. Geck and G. Hiss, Basic sets of Brauer characters of finite groups of Lie type, *J. reine angew. Math.* **418** (1991), 173 - 188. MR **92e:**20006

[Ge] P. Gérardin, Weil representations associated to finite fields, *J. Algebra* **46** (1977), 54 - 101. MR **57:**470

[GMST] R. M. Guralnick, K. Magaard, J. Saxl, and Pham Huu Tiep, Cross characteristic representations of odd characteristic symplectic groups and unitary groups, *J. Algebra* **257** (2002), 291 - 347.

[GPPS] R. M. Guralnick, T. Penttila, C. Praeger, and J. Saxl, Linear groups with orders having certain large prime divisors, *Proc. London Math. Soc.* **78** (1999), 167 - 214. MR **99m:**20113

[GT] R. M. Guralnick and Pham Huu Tiep, Low-dimensional representations of special linear groups in cross characteristic, *Proc. London Math. Soc.* **78** (1999), 116 - 138. MR **2000a:**20016

[HM] G. Hiss and G. Malle, Low dimensional representations of special unitary groups, *J. Algebra* **236** (2001), 745 - 767. MR **2001m:**20019

[Ho] C. Hoffman, Cross characteristic projective representations for some classical groups, *J. Algebra* **229** (2000), 666 - 677. MR **2001f:**20029

[Hw1] R. Howe, On the characters of Weil's representations, *Trans. Amer. Math. Soc.* **177** (1973), 287 - 298. MR **47:**5180

[Hw2] R. Howe, θ-series and invariant theory, *Proc. Symp. Pure Math.* **33** (1979), part 1, pp. 257 - 285. MR **81f:**22034

[Is] I. M. Isaacs, Characters of solvable and symplectic groups, *Amer. J. Math.* **95** (1973), 594 - 635. MR **48:**11270

[JLPW] C. Jansen, K. Lux, R. A. Parker, and R. A. Wilson, '*An ATLAS of Brauer Characters*', Oxford University Press, Oxford, 1995. MR **96k:**20016

[KLi] P. B. Kleidman and M. W. Liebeck, '*The Subgroup Structure of the Finite Classical Groups*', London Math. Soc. Lecture Note Ser. no. 129, Cambridge University Press, 1990. MR **91g:**20001

[Ku] S. Kudla, Seesaw dual reductive pairs, in: '*Automorphic Forms of Several Variables*', Taniguchi Symposium, Katata, 1983, Birkhäuser, Boston, 1983, pp. 244 - 268. MR **86b:**22032

[LaS] V. Landazuri and G. M. Seitz, On the minimal degrees of projective representations of the finite Chevalley groups, *J. Algebra* **32** (1974), 418 - 443. MR **50:**13299

[LST] J. M. Lataille, P. Sin and Pham Huu Tiep, The modulo 2 structure of rank 3 permutation modules for odd characteristic orthogonal groups, *J. Algebra* **268** (2003), 463 - 483.

[Li] M. W. Liebeck, Permutation modules for rank 3 symplectic and orthogonal groups, *J. Algebra* **92** (1985), 9 - 15. MR **86d:**20057

[L] G. Lusztig, '*Characters of Reductive Groups over a Finite Field*', Annals of Math. Studies **107**, Princeton Univ. Press, Princeton, 1984. MR **86j:**20038

[Lu] F. Lübeck, Charaktertafeln für die Gruppen $CSp_6(q)$ mit ungeradem q und $Sp_6(q)$ mit geradem q, Preprint 93-61, IWR Heidelberg, 1993.

[MT1] K. Magaard and Pham Huu Tiep, Irreducible tensor products of representations of finite quasi-simple groups of Lie type, in: '*Modular Representation Theory of Finite Groups*', M. J. Collins, B. J. Parshall, L. L. Scott, eds., Walter de Gruyter, Berlin et al, 2001, pp. 239 - 262. MR **2002m:**20024

[MT2] K. Magaard and Pham Huu Tiep, The classes $\mathcal{C}_6$ and $\mathcal{C}_7$ of maximal subgroups of finite classical groups, (in preparation).

[MMT] K. Magaard, G. Malle, and Pham Huu Tiep, Irreducibility of tensor squares, symmetric squares, and alternating squares, *Pacific J. Math.* **202** (2002), 379 - 427. MR **2002m:**20025

[S] G. M. Seitz, Some representations of classical groups, *J. London Math. Soc.* **10** (1975), 115 - 120. MR **51:**5789

[SS] J. Saxl and G. M. Seitz, Subgroups of algebraic groups containg regular unipotent elements, *J. London Math. Soc.* **55** (1997), 370 - 386. MR **98m:**20057

[SZ] G. M. Seitz and A. E. Zalesskii, On the minimal degrees of projective representations of the finite Chevalley groups, II, *J. Algebra* **158** (1993), 233 - 243. MR **94h:**20017

[ST] P. Sin and Pham Huu Tiep, Rank 3 permutation modules of finite classical groups, (submitted).

[T] Pham Huu Tiep, Dual pairs and low-dimensional representations of finite classical groups, (in preparation).

[TZ1] Pham Huu Tiep and A. E. Zalesskii, Minimal characters of the finite classical groups, *Comm. Algebra* **24** (1996), 2093 - 2167. MR **97f:**20018

[TZ2] Pham Huu Tiep and A. E. Zalesskii, Some characterizations of the Weil representations of the symplectic and unitary groups, *J. Algebra* **192** (1997), 130 - 165. MR **99d:**20074

[Wh1] D. White, Decomposition numbers of $Sp_4(2^a)$ in odd characteristics, *J. Algebra* **177** (1995), 264 - 276. MR **96k:**20023

[Wh2] D. White, Decomposition numbers of unipotent blocks of $Sp_6(2^a)$ in odd characteristics, *J. Algebra* **227** (2000), 172 - 194. MR **2001d:**20012

[Zs] K. Zsigmondy, Zur Theorie der Potenzreste, *Monath. Math. Phys.* **3** (1892), 265 - 284.

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